Lecture 1   Wave and Beam optics

Mostly from Chapters 2, 3, 5

The fundamentals of optics at the introductory level are mostly treated using the "plane wave" model. The plane waves have an infinite extent in the transverse direction. Plane waves, however, never existed in the real world. This lecture is to establish the basics of waves of finite cross section.

Wave function

Light propagates in the form of waves. In free space, the waves are governed by the wave equation,

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0 \qquad (1)$$

where u is a function of **r** and *t*. In a one-dimensional system, the solutions are *u(x-ct) and u(x+ct).*

From the theory of electromagnetics, both the E and H fields of the electromagnetic waves in vacuum satisfies Eq. (1). The form of Eq.(1) for E and H can be derived from Maxwell's equations.

$$\begin{aligned} &\nabla \times H = \varepsilon_0 \frac{\partial E}{\partial t} \\ &\nabla \times E = -\mu_0 \frac{\partial H}{\partial t} \qquad \textit{Maxqwell' sequation in free space} \\ &\nabla \cdot E = 0 \\ &\nabla \cdot H = 0 \end{aligned}$$

Try to prove that both the E and H satisfy equation (1).

The monochromatic harmonic waves can be expressed as

$$U(r,t) = a(r)e^{j\phi(r) - j2\pi\nu t} \qquad (2)$$

where U is known as the complex wave function, and the real part, u(r,t), is given by

$$u(r,t) = U + U^* \qquad (3)$$

Why using the harmonic waves?

The " wavefronts" are the surfaces of equal phase $\phi(r) = cons\tan t$. The optical intensity (power per unit area) is given by $I(r) = |U(r)|^2$.

The Helmhotz Equation: *(governing the spatial dependency of monochromatic waves)*

For a given harmonic function U, the spatial-temporal differential equation in (1) can be reduced to a differential equation in space only. The function U in Eq.(2) satisfies the following equation:

$$(\nabla^2 + k^2)U(r) = 0$$

$$where \ \ k = \frac{2\pi\nu}{c} = \frac{\omega}{c} \tag{4}$$

is the wave number.

Solution 1: plane waves

$$U(r,t) = Ae^{-i\vec{k}\bullet\vec{r} - j2\pi\nu t} \tag{5}$$

where $\boldsymbol{A}$ represents the magnitudes of the electric and magnetic fields. The plane of constant phase is perpendicular to $\boldsymbol{k}$. (This can be easily proved by choosing one of the coordinate axes to be along the direction of $\boldsymbol{k}$. Then the derivatives of A in the orthogonal directions are zero.)

From Maxwell's equations, the vectors of ***E, H*** and ***k*** are orthogonal to each other.

Solution 2: Spherical waves in the outward direction from the origin.

$$U(r,t) = \frac{A}{r} e^{-j\vec{k}\bullet\vec{r} - j2\pi\nu t} \tag{6}$$

Can be proved by expressing the Laplacian in the spherical coordinates.

This is the wave originating from a point and propagating outward with diminishing intensity with the square of the distance.

Solution 3: Fresnel approximation of spherical waves (Paraboloidal waves)

At a point close to the z-axis but far from the origin so that $(x^2 + y^2)^{1/2} << z$

$\Rightarrow$

$$U(r,t) = \frac{A}{z} e^{jkz} e^{-jk\frac{x^2 + y^2}{2z}} \tag{7}$$

Paraxial rays

The plane-wave solution propagating in the z-direction is given by

$$U(r) = Ae^{ikz}$$

If the constant amplitude is allowed to vary slowly,

$$U(r) = A(z)e^{ikz} \tag{8}$$

where $A(z)$ must vary slowly with position z within the distance of a wavelength. This is the expression for the paraxial wave in the z-direction. See picture Figure 2.2-5

Conditions for paraxial waves

$\frac{\partial A}{\partial z} << kA$ where $k = \frac{2\pi}{\lambda}$. (8.1)

$$\frac{\partial^2}{\partial z^2} A << k^2 A$$

In order that *U* satisfies the Helmholtz equation, the complex amplitude must satisfy the following **paraxial Helmholtz equation**:

$$\nabla_T^2 A - 2jk\frac{\partial A}{\partial z} = 0 \tag{9}$$

Where ∇_T^2 denotes the transverse part of the Laplacian. This is the slowly varying envelop approximation for paraxial rays.

One simple solution to the paraxial Helmholtz equation is by assuming that the field variation depends on z and lateral distance only:

$$A(r) = \frac{A_1}{z} e^{-jk\frac{\rho^2}{2z}}. \tag{10}$$

where A_1 is a constant and $\rho^2 = x^2 + y^2$. Eq. (10) is the same as Eq (7). The higher-order field which has an azimuthal dependence will be discussed later.

To prove the , simply insert (10) into (9).

Gaussian beam (Chapter 3)

Another useful solution for the paraxial Helmholtz equation is the Gaussian beam:

$$A(r) = \frac{A_1}{q(z)} e^{[-jk\frac{\rho^2}{2q(z)}]} \qquad q(z) \equiv z - \xi \tag{11}$$

Eq (11) is obtained from Eq (10) with the z axis shifted by a complex constant. A special case when ξ is a imaginary number *(ξ=-jz₀)* is particularly useful:

$$A(r) = \frac{A_1}{q(z)} e^{[-jk\frac{\rho^2}{2q(z)}]} \qquad q(z) \equiv z + jz_0 \tag{12}$$

Now A(r) becomes the complex envelop of the Gaussian beam. The parameter z_0 is the Rayleigh range. (*What is the physical meaning of the Rayleigh range?*)

To separate the real and imaginary parts of the parameter *q(z),* we express it using the following relation:

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi W^2(z)} \tag{13}$$

(The meaning of R and W will be apparent later.)

From Eq.(8), (12), and (13), the wave function for Gaussian beam can be expressed as

$$U(r) = A_0 \frac{W_0}{W(z)} e^{[-\frac{\rho^2}{W^2(z)}]} e^{[-jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)]} \tag{14}$$

where

$W(z) = W_0[1 + (\frac{z}{z_0})^2]^{1/2}$	Beam size at $1/e^2$	(15)
$R(z) = z[1 + (\frac{z_0}{z})^2]$	Radius of curvature	(16)
$\theta_0 = \frac{\lambda}{\pi W_0}$	Divergence angle	(17)
$W_0 = (\frac{\lambda z_0}{\pi})^{1/2}$	Beam waist at minimum	(18)
$\zeta(z) = \tan^{-1}(\frac{z}{z_0})$		(19)

Properties of Gaussian beam

Intensity at any position **r**: From (14),

$$I(\rho, z) = I_0[\frac{W_0}{W(z)}]^2 e^{[-\frac{2\rho^2}{W^2(z)}]} \tag{20}$$

where W*(z)* increases with *z* according to the following relation:

$$W(z) = W_0[1 + (\frac{z}{z_0})^2]^{1/2}$$

Thus the bema expands with increasing z.

The axis is chosen so that the minimum waist occur at z=0. The beam expands and peak intensity decrease with increasing z, according to (20).

Rayleigh range: the distance over which the beam size expands by $\sqrt{2}$ times or the intensity reduces by a factor or two.

Depth of focus --- $$2z_0 = \frac{2\pi W_0^2}{\lambda} \quad (21)$$

Examples: For λ=1.0 μm

W_0 =	z_0 =
1 cm	31400 cm
1 mm	314 cm
100 μm	3 cm

Beam divergence

Comparison with the case of plane waves restricted by an aperture:

$\theta = 1.22\frac{\lambda}{D}$ (half angle of divergence of plane waves of diameter D.

What is the difference in beam profile between Gaussian and truncated plane wave?

For an open aperture of 1 cm and λ=1.0 μm, the divergence, from the center to the first minimum , is 12.2 micro-rad.

For a Gaussian beam of 1-cm waist, the divergence, measured from the center to $1/e^2$ is 29 micro-rad.

Definitions:

Far field $z \gg z_0$

In the far field, the beam profile remains constant and expands linearly with z. The shape of the beam is simple and does not change with distance.

Near field: z on the same order of z_0

The amplitude and phase changes with distance rapidly.

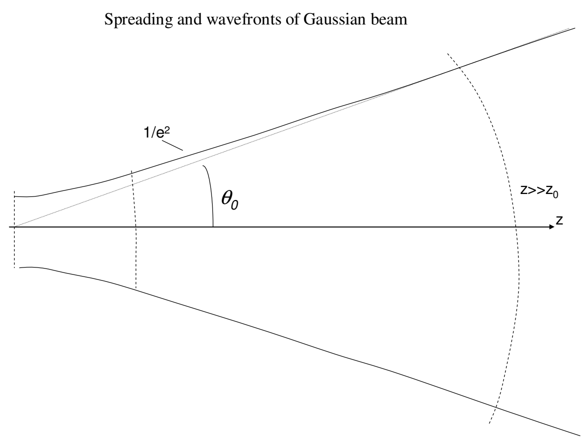
Phase of of Gaussina beam

From Eq.(14), the imaginary part of the exponents is

$$\phi(\rho, z) = kz - \zeta(z) + \frac{k\rho^2}{2R(z)} \quad (22)$$

Along the axis,when ρ=0, the phase has two components. The first component is the phase of plane waves and the second is the correction factor given by (19). The correction is negative. The total accumulated phase deviation from the plane wave by propagating from z=-∞ to z=+∞ is π. For a Gaussian beam, the wave is a superposition of plane waves in various directions, resulting a delay in the z-direction of the composite waves.

The constant phase surface can be illustrated by the following diagram:



Examples of the evolution of the wave fronts at various locations

For plane waves, the $W_0 = \infty$, the spreading is zero.

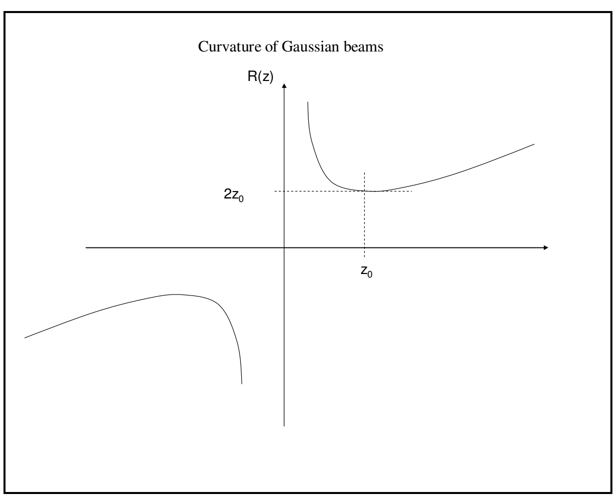
***Radus of Curvature** of Gaussian waves.*

$$R(z) = z[1+(\frac{z_0}{z})^2] \tag{22}$$

For $z=0$, $R=\infty$. For $z=\infty$, $R=\infty$. The minimum of $R(z)$ occurs at $z=\pm z_0$. The value is

$$R(\pm z_0) = 2z_0$$

Thus the wavefront is most curved at the Rayleigh range.



To draw a diagram of wave fronts at various locations.

A Review of the ABCD Law of the ray optics (Chapter 1)

In geometric optics, an arbitrary optical system can be represented by a 2x2 matrix, which transforms the position and angles of the incident beam into the position and angle of the outgoing beam.

$$\begin{bmatrix} r_{out} \\ r'_{out} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_{in} \\ r'_{in} \end{bmatrix} \tag{23}$$

To s*how a diagram*

Common optical systems:

Homogeneous medium of length d $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$ (24)

Thin lens of focal length *f* $\begin{bmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{bmatrix}$ *(25)*

Dielectric interface $\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$ (26)

Spehrical mirror of radius (concave) $\begin{bmatrix} 1 & 0 \\ \frac{-2}{R} & 1 \end{bmatrix}$ (27)

A medium with quadratic index profile $\begin{bmatrix} \cos(\sqrt{\frac{k_2}{k}}l) & \sqrt{\frac{k}{k_2}}\sin(\sqrt{\frac{k_2}{k}}l) \\ -\sqrt{\frac{k_2}{k}}\sin(\sqrt{\frac{k_2}{k}}l) & \cos(\sqrt{\frac{k_2}{k}}l) \end{bmatrix}$ (28)

where the index of refraction varies with the lateral position according to

$n(\rho) = n_0\left[1 - \frac{k_2}{2k}\rho^2\right]$ and l is the length. (29)

Examples of application of the ABCD law to ray optics.

1. A ray of a given position and slope propagating in free space.
2. Snell's law
3. Parallel rays incident on converging lens.
4. Special case for a ray parallel to the axis entering a quarter-pitch, half-pitch graded-index lens.

The ABCD law for the beam optics

The q parameter of the transmitted Gaussian beam through an optics is related to that of the incident beam by the following relations:

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D}$$

Where q_1 is the incident and q_2 is the transmitted and A,B, C and D are the elements of the ABCD matrix.

Example:

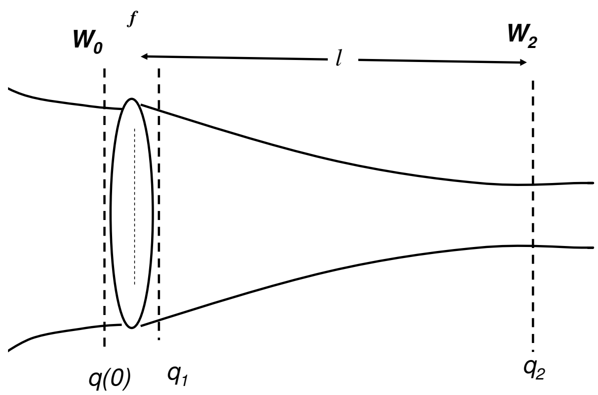
1. A Gaussian beam has a waist of W_0. Find the q-parameter, beam size and radius of curvature after propagating through a uniform medium of distance z.

   The waist at z=0 $\quad \frac{1}{q(0)} = \frac{1}{R(0)} - j\frac{\lambda}{\pi W^2(0)} = -j\frac{\lambda}{\pi W_0^2}$

   From the ABCD law $\quad q(z) = \frac{q_0 + z}{0+1} \qquad (30)$

   q(z) is also defined through Eq. (13)
   By equating the real and imaginary parts of the Eqs (13) and (30), the result should be the same as Eqs.(15) and (16)

2. A lens of focal length f is placed at the waist of a Gaussian beam of waist W_0. Find the location and size of the minimum spot of the transmitted beam.



Approach: the beam is transformed by the lens followed by the propagation in free-space of distance l after the lens. The q-value may be calculated for each step.

The initial q value $\frac{1}{q(0)} = \frac{1}{R(0)} - j\frac{\lambda}{\pi W^2(0)} = -j\frac{\lambda}{\pi W_0^2}$ because the radius of curvature is infinity.

The q-value after the lens, using the ABCD rule, is $q_1 = \frac{q(0)}{-(\frac{q(0)}{f})+1} = \frac{j\frac{\pi W_0^2}{\lambda}}{-j\frac{\pi W_0^2}{\lambda f}+1}$

The q-value at the waist after propagating through a distance l is $q_2 = q_1 + l$.

Set $\frac{1}{q_2} = \frac{1}{R_2} - j\frac{\lambda}{\pi W_2^2}$. By requiring that $R_2 = \infty$ and equating the real and imaginary parts of the equation, the location of the new waist l can be found to be

$$l = \frac{f}{1+(\frac{f}{z_0})^2} \qquad (31)$$

where z_0 is the Rayleigh range for the incident beam. $z_0 = \frac{\pi W_0^2}{\lambda}$

The new beam waist is given by

$$\frac{W_2}{W_0} = \frac{\frac{f}{z_0}}{\sqrt{1+(\frac{f}{z_{01}})^2}} \qquad (32)$$

Discussions:

- For a incident beam of planar wavefront with large waist, the minimum spot occurs at the focal length. (From Eq. (31))
- What is the minimum focused spot size using a length of focal length *f?*
- *How does the spot size of the focused beam depend of the size of the incident beam?*

Higher-order Gaussian beam

If we do not impose the condition of azimuthal invariance in the Helmholtz equation , then the solutions can be expressed in the form of Hermite polynomials.

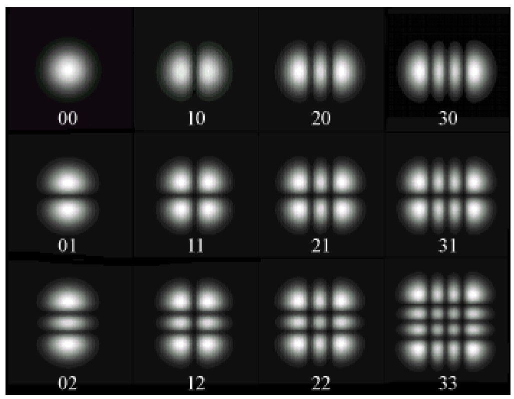
$$U_{l,m}(x,y) = A_1 \frac{W_0}{W(z)} H_l(\sqrt{2}\frac{x}{W(z)}) H_m(\sqrt{2}\frac{y}{W(z)}) \times$$

$$e^{[-\frac{\rho^2}{W^2(z)} - \frac{jk\rho^2}{2R(z)} - jkz + j(l+m+1)\zeta]}$$

(33)

Where H_l and H_m *are* the Hermite polynomial of order *l* and *m.* The rest of the parameters are defined the same as for the fundamental Gaussian mode.

The phase shift along the axis is the larger. The transverse variation follows the Gaussian envelop function with a modulation defined by H_l and $H_{m.}$



HOMEWORK

1. Starting from Maxwell equations, prove that the electric and magnetic fields in free space follows the wave equation Eq.(1). List all the assumptions and how the speed of the light is related to vacuum susceptibility and permeability. The derivation can be found in any book in the Chapter of electromagnetic waves. You need to do the derivation once.

2. A lens of focal length f is placed at the waist of a Gaussian beam of waist W_0. Find the location and size of the minimum spot of the transmitted beam. Determine the minimum spot size of the focused beam.

3. Problem 1.4.2:
   Ray-Transfer Matrix of a GRIN Plate. Determine the ray-transfer matrix of a SELFOC plate [i.e., a graded-index material with parabolic refractive index $n(y) \approx n_0(1-\frac{1}{2}\alpha^2 y^2)$] of width *d*.
   Problem 1.4.3:
   The GRIN Plate as a Periodic System. Consider the trajectories of paraxial rays inside a SELFOC plate normal to the *z* axis. This system may be regarded as a periodic system made of a sequence of identical continuous plates of thickness *d* each. Using the result of Problem 1.4.2, determine the stability condition of the ray trajectory. Is this condition dependent on the choice of *d*?

4. Problem 3.1.1:
   Beam Parameters. The light from a Nd:YAG laser at wavelength 1.06 μ m is at a Gaussian beam of 1-W optical power and beam divergence $2\theta_0 = 1$ mrad. Determine the beam waist radius, the depth of focus, the maximum intensity, and the intensity on the beam axis at a distance $z = 100$ cm from the beam waist.

   Problem 3.1.2:
   Beam Identification by Two Widths. A Gaussian beam of wavelength $\lambda_0 = 10.6\mu m$ (emitted by a CO_2 laser)has widths $W_1 = 1.699mm$ and $W_2 = 3.38mm$ at two points separated by a distance $d = 10cm$. Determine the location of the waist and the waist radius.

   Problem 3.2.1:
   Beam Focusing. An argon-ion laser produces a Gaussian beam of wavelength $\lambda = 488nm$ and waist radius $W_0 = 0.5mm$. Design a single-lens optical system for focusing the light to a spot of diameter 100 μm .What is the shortest focal-length lens that may be used?

5. Problem3.2.4:
   Transmission of a Gaussian Beam Through a Graded-Index Slab. The *ABCD* matrix of a SELFOC graded-index slab with quadratic refractive index (see Sec.1.3B) $n(y) \approx n_0(1-\frac{1}{2}\alpha^2 y^2)$ and length *d* is : $A = \cos\alpha d$, $B = (1/\alpha)\sin\alpha d$, $C = -\alpha\sin\alpha d$, $D = \cos\alpha d$ for paraxial rays along the z direction. A Gaussian beam of wavelength λ_0 ,waist radius W_0 in free space, and axis in the z direction enters the slab at its waist. Use the *ABCD* law to determine an expression for the beam width in the y direction as a function of *d*. Sketch the shape of the beam as it travels through the medium.

6. AGaussian beam of wavelength λ is incident on a lens placed at $z = l$ as shown. Calculate the lens focal length *f* so that the output beam has a waist at the front surface of the sample crystal. Show that(given *l* and L) up to two solutions may exist. Sketch the beam behavior for each of these solutions.