2 Waves in dielectric media and waveguides

Section 5.2

In this lecture, we will consider the properties of waves whose propagation is governed by both the diffraction and confinement processes. The waveguides are a result of the balance between the the diffraction and confinement.

The concept of wave propagation as a re-emission process.

Waves in dielectric media

$$E \longrightarrow P = \varepsilon_0\, \chi\ (r,\ t,\ \theta,\ E)\ E$$

Waveguiding: $\chi\ (r\ )$

Dispersive: $\chi\ (\ t\ )$ or $\chi\ (v)$

Nonlinear: $\chi\ (\ E)$

$$P(t) = \int_{-\infty}^{t} \chi(t - t') E(t') dt'$$

Wave equation in linear non-dispersive homogenous and isotropic media

Non-dispersive media-- here it means " instantaneous".

The vector of the polarization density is instantaneously following that of the electric field.

$$P = \varepsilon_0 \chi E, \tag{1}$$

Where χ is the electric susceptibility which does not depends on r, t or E. From the relation $D = \varepsilon_0 E + P$ between the electric displacement and polarization, and $D = \varepsilon E$

The dielectric constant is given by

$$\varepsilon = \varepsilon_0 (1 + \chi) \tag{2}$$

Maxwell's equations are

$$\nabla \times H = \frac{\partial D}{\partial t} \quad = \varepsilon \frac{\partial E}{\partial t} \tag{3}$$

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t} \tag{4}$$

$$\nabla \bullet E = 0 \tag{5}$$

$$\nabla \bullet H = 0 \tag{6}$$

Which lead to the same form of wave equation as in free space except that the speed of light is redefined.

$$\nabla^2 U - \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = 0$$

Where $c = \dfrac{c_0}{n} = \dfrac{c_0}{\sqrt{(1+\chi)}} = \dfrac{c_0}{\sqrt{\varepsilon}}$

Next, we will work out a more general approach to allow P to depend on E.

Waves in a Nonlinear Medium
What is a nonlinear medium?— P is a function of E

With the nonlinear dependency, P cannot be factored into E.

(3) ------ $\nabla \times H = \dfrac{\partial D}{\partial t}$

(4)-------- $\nabla \times \nabla \times E = -\mu_0 \dfrac{\partial \nabla \times H}{\partial t} = -\mu_0 \dfrac{\partial^2 D}{\partial t^3} = -\mu_0 \varepsilon_0 \dfrac{\partial^2 E}{\partial t^2} - \mu_0 \dfrac{\partial^2 P}{\partial t^2}$

It follows that

$$\nabla^2 E - \frac{1}{c_0^2} \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2} \qquad\qquad (7)$$

where P is a nonlinear function of E . P can be at a different frequency from that of E. This equation will be used extensively later. The polarization vector becomes the source which plays a major role in nonlinear frequency generation, laser, and waveguide coupling to be discussed later.

Examples. The *P* may contain multiple powers of *E*. As a result, new frequencies may be generated.

Monochromatic electromagnetic field in a medium

When both *E* and *H* are harmonic waves of frequency ω, Maxwell's equations for a non-dispersive medium become

$\nabla \times B = j\omega D$

$\nabla \times E = -j\omega B$

$\nabla \cdot D = 0$

$\nabla \cdot B = 0$

Where $D = \varepsilon_0 E + P$ and $B = \mu_0 H$. These relations will lead to the Helmholtz equation

$\nabla^2 U + k^2 U = 0 \qquad\qquad (8)$

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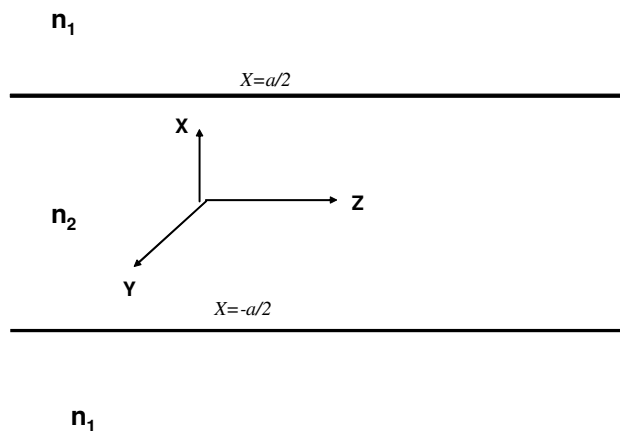where k the wave number in the medium

$$k = k_0 \sqrt{\varepsilon} = k_0 n$$

When the index of refraction is a function of position, then

$$\nabla^2 U + k(r)^2 U = 0 \qquad\qquad (9)$$

Guided modes in a symmetric one-dimensional waveguide

To give a physical pictures of waveguide and waveguide modes using the concepts of waves bouncing between two parallel interfaces:

One-dimensional slab waveguide

n₁

X=a/2

X

n₂

Z

Y

X=-a/2

n₁

For a wave propagating in the z-direction, the solution of the Helmholtz equation (8) may be simplified by assuming that the electric field is a harmonic wave given by

$$E(r) = E(x)e^{j\beta z}$$

The Helmholtz equation can be separately listed for each region.

Region 1:
$$\frac{\partial^2}{\partial x^2}E(x)+(k^2n_1{}^2-\beta^2)E=0 \tag{10}$$

Region 2
$$\frac{\partial^2}{\partial x^2}E(x)+(k^2n_2{}^2-\beta^2)E=0 \tag{11}$$

Region 3
$$\frac{\partial^2}{\partial x^2}E(x)+(k^2n_1{}^2-\beta^2)E=0 \tag{12}$$

The field at the boundary must satisfy the boundary condition at the interfaces., namely the tangential component of E and H to be continuous across the boundary.

Consider the TE mode electric field parallel to the plane, a symmetric solution of the following form

$E_y = C_1 e^{-qx}$ for $x > a/2$ and $x < -a/2$

$E_y = C_2 \cos px$ for $-a/2 < x < a/2$

Where p and h satisfies the following relation
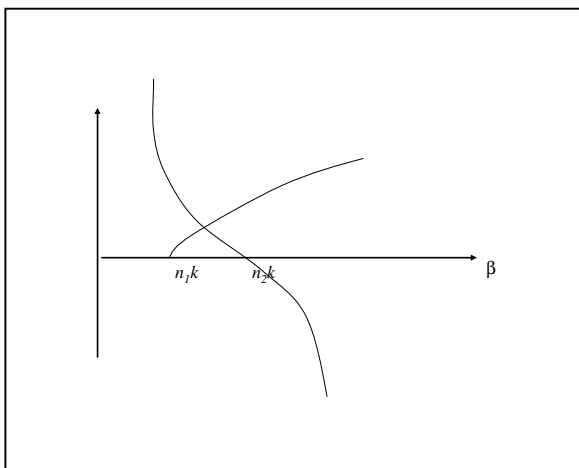
$$q^2 = \beta^2 - n_1^2 k^2 \qquad \tag{13}$$
$$p^2 = n_2^2 k^2 - \beta^2$$

In order to have a guided mode, both p and q must be real numbers so that the waves do not propagates in the x direction, the modal propagating constant satisfies the following relation:
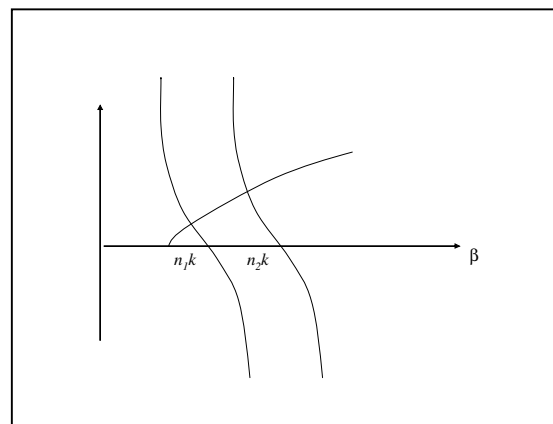
$$n_2 k > \beta > n_1 k \tag{14}$$

The boundary conditions lead to the following equation for the eigen value β.

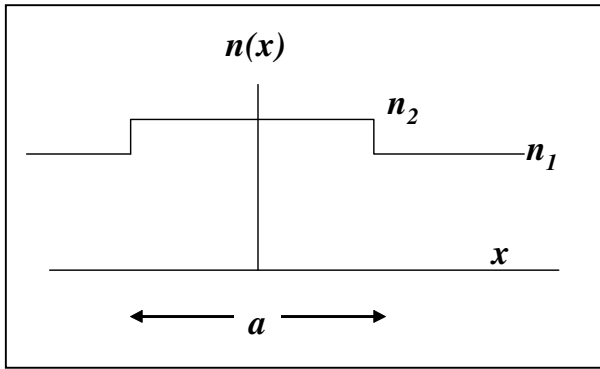$$\tan\frac{\sqrt{n_2^2 k^2 - \beta^2}\,a}{2} = \sqrt{\beta^2 - n_1^2 k^2} \tag{15}$$
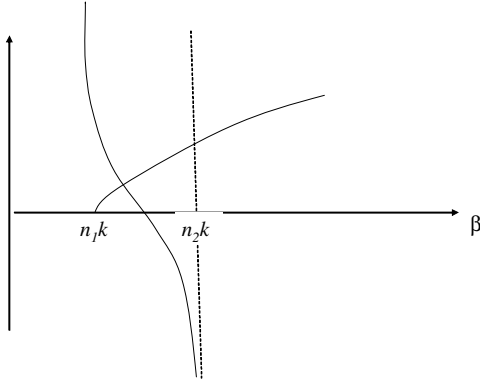

Single mode

Multimode

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The symmetric waveguide always has a solution. Depending on the steps of the refractive indices, there can be more than one mode.

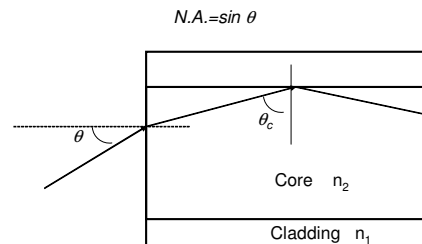From the figure for the two-mode situation,

$$\frac{\sqrt{n_2^2 k^2 - n_1^2 k^2}\, a}{2} < \pi$$

$$\sqrt{n_2^2 - n_1^2}\, ka < 2\pi \qquad \text{or} \qquad \sqrt{n_2^2 - n_1^2} < \frac{\lambda}{a} \qquad\qquad (16)$$

The factor $\sqrt{n_2^2 - n_1^2}$ is also the numerical aperture of the waveguide.
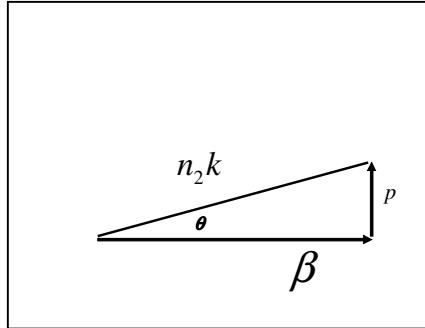
The numerical aperture of waveguide

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The number of modes that can be supported by a waveguide is

$$M = \frac{\sqrt{n_2^2 - n_1^2}\, ka}{2\pi} = N.A. \frac{a}{\lambda}$$

For a silica single mode fiber of core diameter 5 microns and NA=0.1,. Find the index step.

Beam profiles

From Eq. (13)
$$p^2 = n_2^2 k^2 - \beta^2 \quad \text{or} \quad n_2^2 k^2 = \beta^2 + p^2$$

Internal field distribution

The beam profiles of the guided modes can be understood as being the interference fringes of the two waves bouncing between the two interfaces.

The lowest order mode has the smallest p value and no node. The higher order mode has a larger p values and large number of nodes.

Field in the cladding

The field in the cladding decrease exponentially with increasing distance from the core. The larger q value, the faster the decay. The propagation constant β is bound within the range $n_2 k > \beta > n_1 k$. The lowest order mode has the largest value for q. The electric field is more confined in the core. The higher-order mode have smaller values for q and the electric field extends more into the cladding. *This may be counter intuition—why?*

Group velocity of guided modes

The group velocity, the velocity of energy or wave packet that is detectable, is given by
$$Group\,velocity\ v_g = \frac{\partial \omega}{\partial \beta}$$

From Eq. (13), $\quad p^2 = n_2^2 k^2 - \beta^2 = n_2^2 \dfrac{\omega^2}{c^2} - \beta^2$

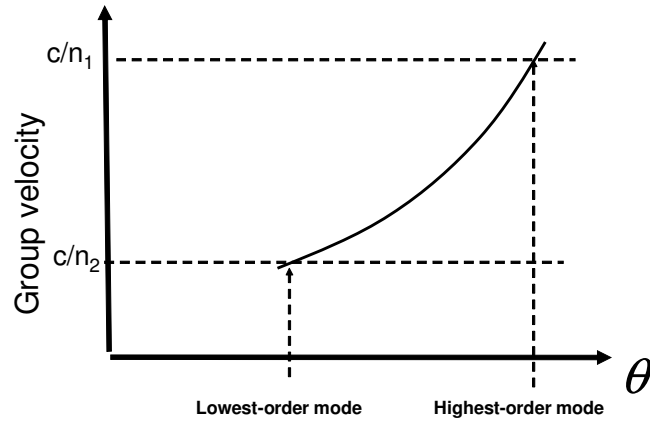We can express ω in terms of β and p in terms of $\beta \tan\theta$.

$$\omega = \frac{c}{n_2} \beta \sec\theta$$

Where θ is the oblique angle between p and β. The lowest order mode has the smallest θ. The group velocity is given by

$$v_g = \frac{c}{n_2} \sec\theta$$

Thus the group velocity of the higher-order modes with a larger θ is larger.

This counter intuition. **Why?**

Waveguide with quadratic refractive index distribution

Returning to Eq. (8)

$$\nabla^2 U + k^2 U = 0 \tag{8}$$

where k the wave number in the medium
$$k = k_0 \sqrt{\varepsilon} = k_0 n$$
When the index of refraction is a function of position, then

$$\nabla^2 U + k(r)^2 U = 0 \tag{9}$$

$$\nabla^2 U + \frac{\varepsilon(\vec{r})\omega^2}{c^2} U = 0$$

The solutions are the eigen modes of the waveguide. If the wave propagates in the z-direction
$$U = u(x)e^{i\beta z}$$

$$\nabla_T u - (\beta^2 - \frac{\omega^2}{c^2}\varepsilon(x))u = 0$$

$$\text{For } n = n_0 - \Delta n\frac{x^2}{a^2} \quad \varepsilon(x) == n^2 \approx n_0^2 - 2\Delta nn_0\frac{x^2}{a^2}$$

$$\frac{d^2u}{dx^2} + [n_0^2\frac{\omega^2}{c^2} - \frac{2\omega^2}{c^2}\Delta nn_0\frac{x^2}{a^2} - \beta^2)u = 0$$

$$\text{Try } u(x) = e^{\alpha x^2} \quad where \; \alpha \, is \, a \, cons$$

$$4\alpha^2 - \frac{2\omega^2}{c^2}\frac{\Delta n}{a^2}n_0 = 0 \Rightarrow \alpha = \pm\frac{1}{\sqrt{2}}\frac{\omega}{c}\frac{\sqrt{\Delta n \, n_0}}{a}$$

$$\beta^2 = k_0^2 + 2\alpha$$

Case I Δn= a real number

$$\beta = \pm k_0(1 + \frac{\alpha}{k_0^2}) = \pm k_0 \pm \frac{1}{\sqrt{2}a}\sqrt{\frac{\Delta n}{n_0}}$$

The electric field is

$$E = E_0 e^{\pm j(k_0 \pm \frac{1}{\sqrt{a}}\sqrt{\frac{\Delta n}{n_0}})z} \; e^{\pm\frac{1}{\sqrt{2}}k_0\sqrt{\frac{\Delta n}{n_0}}\frac{x^2}{a}}$$

Wavefront is planar with a Gaussian distribution. Beam waist at 1/e is proportional to the fourth root of Δn.

Case II Δn= a pure imaginary number jΔn_i (gain guide)

It is customary to express the spatial variation in terms of the real and imaginary part of the refractive index.

$$E = E_0 e^{\pm j(k_0 \pm \frac{1}{\sqrt{a}}\sqrt{\frac{\Delta n_i}{n_0}})z} \; e^{\pm k_0\sqrt{\frac{\Delta n_i}{n_0}}\frac{x^2}{a} \pm jk_0\sqrt{\frac{\Delta n_i}{n_0}}\frac{x^2}{a}}$$

The waveguide supported by the imaginary part of the refractive index is known as the gain-guide. The power distribution is also Gaussian. The wavefront is a curved surface governed by

$$\pm(k_0 \pm \frac{1}{\sqrt{a}}\sqrt{\frac{\Delta n_i}{n_0}})z \pm k_0\sqrt{\frac{\Delta n_i}{n_0}}x^2 = cons\tan t.$$

The wavefront is a cylindrical surface.

Problems:

1. For an optical fiber of core diameter 5 microns, estimate the difference in index refraction between the core and the cladding to support a single mode for λ=1-μm. How many modes can this waveguide support for λ=0.5 μm.?

2..Prove that the numerical aperture, N.A, of a waveguide is given by $\sqrt{n_2^2 - n_1^2}$.

3. Find the maximum disparity in group velocity of the different modes in a waveguide. Assuming that the lowest possible oblique angle is 0 and the maximum angle is determined by the total internal reflection between the core and cladding.

4. Two waves of wavelength λ and propagating in k_1 and k_2 directions overlap in space. The interference between the waves results in stationary fringes.

Find the fringe spacing and the direction of the fringe lines in relation k_1 and k_2.
Sketch the fringe lines.
Discuss the limiting cases when $k_1 = k_2$ and $k_1 = - k_2$..
Try to use the relation for fringe spacing to explain the fringe spacing of the double slit (Young's) experiment.