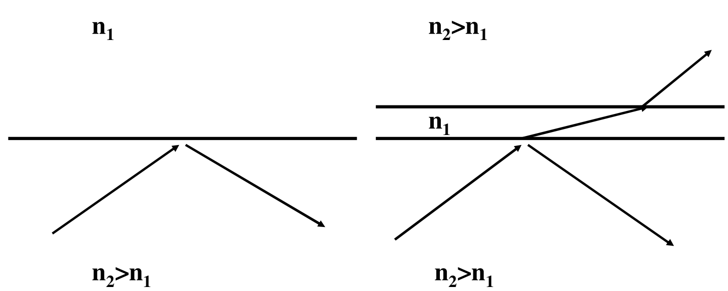
## 3. Theory of coupled waveguides and coupled mode analysis

*Some of the concepts discussed in this lecture may be applicable to the interaction of multiple waves.*

The "vanishing" wave extending beyond the interface of total internal reflection is called the evanescent waves.

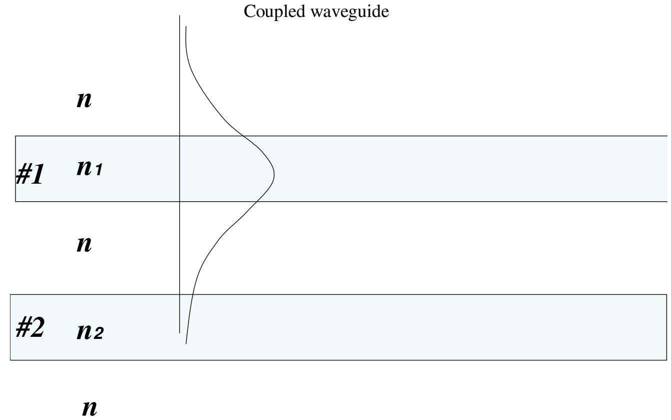
The evanescent waves may become propagating waves again if a second medium of larger index of refraction is placed in proximity to the boundary.

Evanescent waves




*Evanescent coupling* .

Coupled waveguide




For waveguides, **without the coupling**, the waves can be expressed in the following form

$$E_1(x,z,t) = a_1 u_1(x) e^{-j\beta_1 z} e^{j\omega t} \qquad (1)$$

$$E_2(x,z,t) = a_2 u_2(x) e^{-j\beta_2 z} e^{j\omega t} \quad (2)$$

Where $a_1$ and $a_2$ are constants. The electric fields, $E_1$ and $E_2$, satisfies the Helmholtz equation:

$$\nabla^2 E_1 + \frac{1}{c_1^2}\frac{\partial^2 E_1}{\partial t^2} = 0 \quad \textbf{\textit{(2.1)}}$$

$$\nabla^2 E_2 + \frac{1}{c_2^2}\frac{\partial^2 E_2}{\partial t^2} = 0 \quad \textbf{\textit{(2.2)}}$$

When the wave of one waveguide extends into the second, the electric field of the first wave produces a polarization in the second to act as a source. The polarization term is no longer time independent. The following is how to treat the effect of the second on the first through perturbation.

The presence of the second waveguide creates a change in the index of refraction ($n_2$-n) at the position of the second waveguide. The polarization created by the refractive index change and the electric filed $E_2$ is

$$P_1 = (n_2^2 - n^2)E_2 \quad (2.3)$$

Likewise

$$P_2 = (n_1^2 - n^2)E_1 \quad (2.4)$$

The Helmholtz equation with a source, caused by the time dependent *P*, ( from Eq. (7) in Lecture 2 )

$$\nabla^2 E_1 - \frac{1}{c_1^2}\frac{\partial^2 E_1}{\partial t^2} = \mu_0 \frac{\partial^2 P_1}{\partial t^2} \quad \textbf{(3)}$$

$$\nabla^2 E_2 - \frac{1}{c_2^2}\frac{\partial^2 E_2}{\partial t^2} = \mu_0 \frac{\partial^2 P_2}{\partial t^2} \quad \textbf{(4)}$$

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Assuming a wave function in the following form

$$E_1(x,z,t) = a_1(z) u_1(x) e^{-j\beta_1 z} e^{j\omega t} \quad (5)$$

$$E_2(x,z,t) = a_2(z) u_2(x) e^{-j\beta_2 z} e^{j\omega t} \quad (6)$$

We assume that the *a* coefficients are now functions of z and the functions u are still the solutions of Eq. (2.1) and (2.2).

The Helmholtz equations for the coupled waveguides can be written as, from (3), (4), (2.3) and (2.4)

$$\nabla^2 E_1 + k_1^2 E_1 = -(k_2^2 - k^2)E_2 \quad \textbf{(7)}$$

$$\nabla^2 E_2 + k_2^2 E_2 = -(k_2^2 - k^2)E_1 \quad \textbf{(8)}$$

Substituting (5) and (6) into (7) and (8), and assuming that $a_1(z)$ and $a_2(z)$ are slow-varying functions of $z$ in the limit of weak coupling, the following relations can be obtained.

$$2\frac{da_1}{dz}u_1 = -(k_2^2 - k^2)a_2u_2e^{j\Delta\beta z} \qquad (9)$$

$$2\frac{da_2}{dz}u_2 = -(k_2^2 - k^2)a_2u_2e^{-j\Delta\beta z}$$

Multiplying both sides by $u_1(x)$ and integrating with respect to x:

$$\frac{da_1}{dz} = -jC_{21}a_2(z)e^{j\Delta\beta z} \qquad (10)$$

$$\frac{da_2}{dz} = -jC_{12}a_1e^{-j\Delta\beta z} \qquad (11)$$

where the coupling strengths C's are defined as

$$C_{21} = \frac{1}{2}\int_{-\infty}^{\infty}(k_2^2 - k^2)u_1u_2dx \qquad (12)$$

$$C_{12} = \frac{1}{2}\int_{-\infty}^{\infty}(k_1^2 - k^2)u_1u_2dx \qquad (13)$$

and $\Delta\beta = \beta_1 - \beta_2$ is the mismatch of the wave vectors for the two waveguide.

The solutions for an initial condition of waves entering waveguide #1 and nothing in waveguide #2, and equal coupling coefficients.

$$a_1(z) = a_1(0)\exp(\frac{j\Delta\beta\, z}{2})(\cos\gamma z - j\frac{\Delta\beta}{2\gamma}\sin\gamma z)$$

$$a_2(z) = a_1(0)\frac{C_{12}}{j\gamma}\exp(-j\frac{\Delta\beta z}{2})\sin\gamma z \qquad (14)$$

where

$$\gamma^2 = \left(\frac{\Delta\beta}{2}\right)^2 + C^2 \text{ and } C = \sqrt{C_{12}C_{21}} \qquad (15)$$

The power is exchanged between the two waveguide following a sinusoidal dependency on $z$ with a period $2\pi/\gamma$. If the two waveguides are identical, a complete periodic extinction can occur.

For a given interaction length, a mismatch can affect the transfer ratio.

**Discussion of applications:**

Wave splitters, combiners, and switches (in conjunction with electro-optic effects).

**Theory of supermodes:**
*Two identical elements*

$$\frac{da_1}{dz} = -\frac{j}{2} C\, a_2(z)$$
$$\frac{da_2}{dz} = -\frac{j}{2} C\, a_1(z) \qquad (16)$$

Try a solution of the following form:

$$a_{1.}(z) = a_1 \exp[-j(\beta_0 + \Delta\beta_c)z)$$
$$a_2(z) = a_2 \exp(-j(\beta_0 + \Delta\beta_c)z) \qquad \textbf{(17)}$$

where $\beta_0$ is the propagating constant for the individual in the absence of the perturbation and $\Delta\beta_c$ is the small changes in the propagation constant.

It can be shown that the propagation constants of the eigen modes of this system are $\beta \pm C/2$, and the wave functions have an amplitude $a_1 = \pm\, a_2$, corresponding to the symmetric and antisymmetric modes.

*N-element supermode*

Eq (16) may be extended to have N waveguides in the system. By generalizing (9), we have

$$\frac{d\vec{a}}{dz} = -j \overleftrightarrow{C} \vec{a} \qquad (18)$$

where *a* is a vector ( or a column of N elements) and C is an *N* x *N* matrix with off-diagonal elements. Assuming that the interaction is between adjacent elements only, the only noz-zero elements are the ones next to the diagonal elements:

$$C_{l,l+1} = \frac{1}{2} \int_{-\infty}^{\infty} (k_{l+1}^2 - k^2) u_l u_{l+1} dx \qquad (19)$$

$$C_{l-1,l} = \frac{1}{2} \int_{-\infty}^{\infty} (k_{l-1}^2 - k^2) u_l u_{l-1} dx \qquad (20)$$

*Special case*: Identical waveguides of equal spacing. All the off-diagonal elements are equal. The *C* matrix has the following form:

$$
\begin{array}{cccccccc}
0 & C & 0 & 0 & & & & \\
C & 0 & C & 0 & & \mathbf{0} & & \\
0 & C & 0 & C & & & & \\
0 & 0 & C & 0 & & & & \\
0 & C & 0 & 0 & 0 & C & 0 & 0 \\
 & & & & C & 0 & C & 0 \\
 & \mathbf{0} & & & 0 & C & 0 & C \\
 & & & & 0 & 0 & C & 0
\end{array}
$$

By assuming the following form for the eigen solution of (18)

$$\vec{a}(z) = \vec{a} \exp[-j\beta_c z] \qquad \textbf{(21)}$$

where the elements of $\vec{a}$ include ($a_1, a_2 ......a_N$), and $\beta_c$ is the eigen value. The eigen value is the solution of the matrix equations.

$$\begin{vmatrix} \beta_c & C & 0 & 0 \\ C & \beta_c & C & 0 \\ 0 & C & \beta_c & C \\ 0 & 0 & C & \beta_c \end{vmatrix} = 0 \qquad (21.1)$$

For a N-element system, the solutions for $\beta_c$ and $\vec{a}$ are

$$a_i^{\ m} = \sin(i \frac{m\pi}{N+1}) \qquad (22)$$

$$\beta_c^{\ m} = C\, con(\frac{\pi m}{N+1}) \qquad (23)$$

where $i$ denote the element number from 1 to N, and $m$ is the order of the mode from 0 to N-1.

Finally the full wave function for the two identical waveguides are

$$E_i^m(x,z,t) = \sin(i \frac{m\pi}{N+1}) u_i(x) e^{-j(\beta+\beta_c)z} e^{j\omega t} \qquad (24)$$

Note that the propagation constant is $\beta + \beta_c$.

Problems:

1. Prove that the eigen value $\beta_c$ and eigen vector $\vec{a}$ for a system of N equally spaced identical waveguides with nearest neighbor coupling is given by (22) and (23).
2. Use the result of (1) to express the various eigen modes and eigen propagation constant $\beta+\beta_c$ for the 2- and 3-element coupled waveguide, where $\beta$ is the propagation constant of a single waveguide in the absence of coupling. Sketch the amplitude of the various modes.