Solutions for Problem Set #3

1. Prove that the eigen value β_c and eigen vector $\vec{a}$ for a system of N equally spaced identical waveguides with nearest neighbor coupling is given by (22) and (23).
2. Use the result of (1) to express the various eigen modes and eigen propagation constant $\beta+\beta_c$ for the 2-, 3-, and 4-element coupled waveguide, where β is the propagation constant of a single waveguide in the absence of coupling. Sketch the amplitude of the various modes

By assuming the following form for the eigen solution

$$\vec{a}(z) = \vec{a}\exp[-j\beta_c z]$$

The system of N coupled waveguides can be described by

$$\begin{bmatrix} \begin{matrix} \beta_c & 0 & 0 & 0 \\ 0 & \beta_c & 0 & 0 \\ 0 & 0 & \beta_c & 0 \\ 0 & 0 & 0 & \beta_c \end{matrix} & 0 \\ 0 & \begin{matrix} \beta_c & 0 & 0 & 0 \\ 0 & \beta_c & 0 & 0 \\ 0 & 0 & \beta_c & 0 \\ 0 & 0 & 0 & \beta_c \end{matrix} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \begin{matrix} 0 & C & 0 & 0 \\ C & 0 & C & 0 \\ 0 & C & 0 & C \\ 0 & 0 & C & 0 \end{matrix} & 0 \\ 0 & \begin{matrix} 0 & C & 0 & 0 \\ C & 0 & C & 0 \\ 0 & C & 0 & C \\ 0 & 0 & C & 0 \end{matrix} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

The eigen values are the solution of the determinant of the following equation:

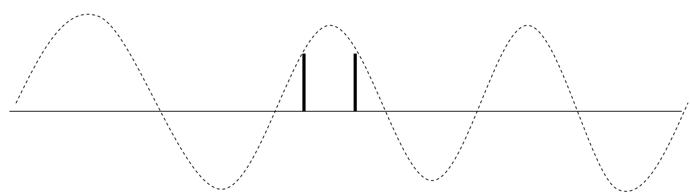
$$\det \begin{bmatrix} \begin{matrix} \beta_c & C & 0 & 0 \\ C & \beta_c & C & 0 \\ 0 & C & \beta_c & C \\ 0 & 0 & C & \beta_c \end{matrix} & 0 \\ 0 & \begin{matrix} \beta_c & C & 0 & 0 \\ C & \beta_c & C & 0 \\ 0 & C & \beta_c & C \\ 0 & 0 & C & \beta_c \end{matrix} \end{bmatrix} = 0$$

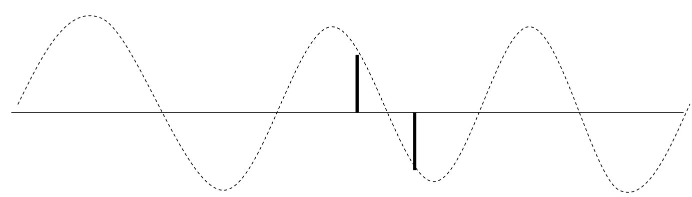
For a 2-element array,

$\beta_c = \pm C$

The amplitudes are $a_1 = \pm a_2$ or or in the general format

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sin\frac{m\pi}{3} \\ \sin\frac{2m\pi}{3} \end{bmatrix}$$ where m is 1 or 2.



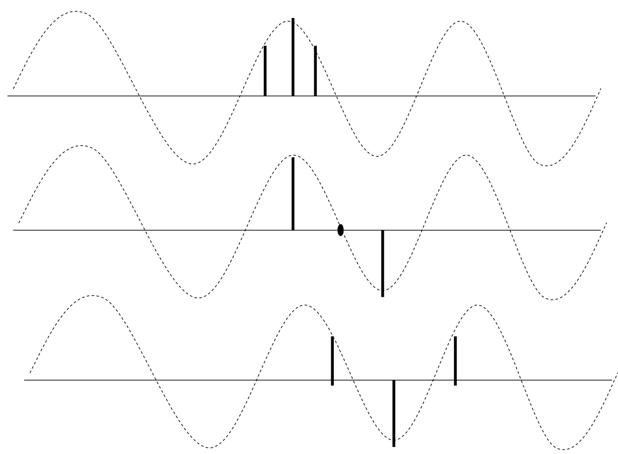


In a three-element system, the solutions by solving the determinant are

$\beta_c = c\ \cos(\frac{m\pi}{4})$ where m is an integer: 1 2, or 3

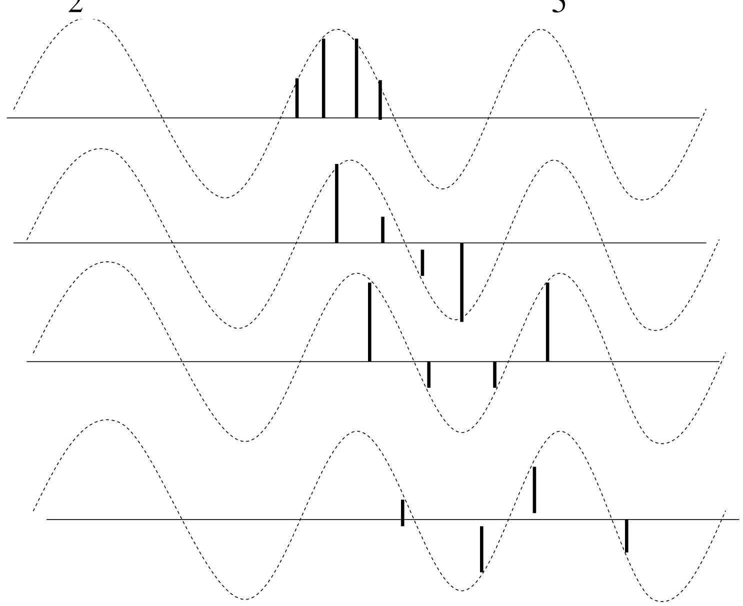
The amplitudes of the wave functions are

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \sin\frac{m\pi}{4} \\ \sin\frac{2m\pi}{4} \\ \sin\frac{3m\pi}{4} \end{bmatrix}$$ where m=1,2 and 3.



In a four element system,

$\beta_c = \frac{\pm 1 \pm \sqrt{5}}{2} C$ and can be expressed as $\cos \frac{m\pi}{5}$ where the m's are integers from 1 to 4.



To prove that the general solution for the eigen values and functions, the governing equations are

$-\beta^m_{_c c} a^m_1 = ca^m_2$

$-\beta^m_c a^m_2 = ca^m_1 + ca^m_3$

•

• where *i* is an integer ranging from 1 to N.

$-\beta^m_c a^m_i = ca^m_{i-1} + ca^m_{i+1}$

•

•

It can be proved that

$\beta^m_c = -2c\cos(\frac{m\pi}{N+1})$ and $a^m_i = \sin(\frac{im\pi}{N+!})$ satisfy all the equations linking *i* and *i+1* and *i-1.*

I have not yet worked out a direct solution to the N x N determinant equation. Let me know if you have figured out how to do.