4 Fourier optics

Fourier relations in Optics

Near field   Far field

Frequency   Pulse duration

Frequency   Coherence length

Beam waist   Beam divergence

Spatial dimension   Angular dimension

Focal plane of lens   The other focal plane

Review of Fourier theorem

A complex function $f(t)$ may be decomposed as a superposition integral of harmonic function of all frequencies and complex amplitude

$$f(t) = \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}dv \quad \text{(inverse Fourier transform)} \tag{1}$$

The component with frequency ν has a complex amplitude $F(\nu)$, given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \quad \text{(Fourier transform)} \tag{2}$$

Selected function and their Fourier transforms, between t and ν, and between x and θ.

$$f(t) = 1 \qquad\qquad\qquad\qquad F(\omega) = \delta(\omega) \tag{3}$$

$$f(t) = \delta(t) \qquad\qquad\qquad\qquad F(\omega) = 1 \tag{4}$$

$$f(t) = \begin{array}{ll} 1 & -\tau/2 < t < \tau/2 \\ 0 & elsewhere \end{array} \qquad\qquad F(\omega) = 2\tau\frac{\sin(\omega\tau/2)}{\omega\tau} \tag{5}$$
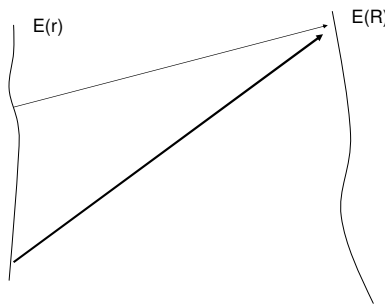
(sinc function)

$$f(t) = \cos(\omega_0 t) \qquad -\frac{\tau}{2} < t < \frac{\tau}{2}$$
$$0 \qquad \text{elsewhere}$$
$$F(\omega) = \tau \frac{\sin[(\omega - \omega_0)\frac{\tau}{2}]}{[(\omega - \omega_0)\frac{\tau}{2}]} \qquad (6)$$

$$f(t) = e^{-\frac{t^2}{\tau^2}} \qquad\qquad F(\omega) = e^{-\tau^2 \omega^2} \qquad (7)$$

$$f(t) = 1 \qquad -\frac{\tau}{2} + nT < t < \frac{\tau}{2} + nT$$
$$n = 0,1, \ldots N-1$$
$$0 \qquad \text{elsewhere}$$
$$F(\omega) = \frac{\sin(\omega\frac{\tau}{2})}{\omega\frac{\tau}{2}} \frac{\sin(\omega\frac{NT}{2})}{\sin(\omega\frac{T}{2})} \qquad (8)$$

Huygens' principle-

Huygens' Principle

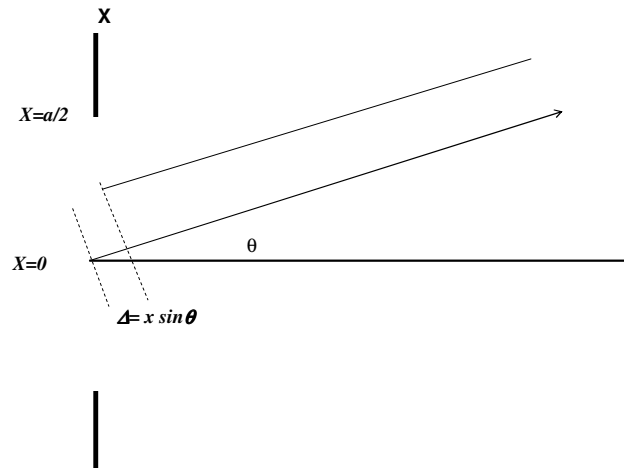$$E(\vec{R}) = \int \frac{E(\vec{r})e^{j\vec{k}\cdot(\vec{R}-\vec{r})}}{|\vec{R} - \vec{r}|} d\vec{r} \qquad\qquad (10$$

Applications of Eq. (5)

Single- slit diffraction

When a single slit of width a illuminated by radiation of wavelength λ, the angular distribution of electric field observed at infinity is

$$E(\theta) = \int_{-\infty}^{\infty} e^{ikx\sin\theta} dx = a \frac{\sin(k\frac{a}{2}\sin\theta)}{k\frac{a}{2}\sin\theta} \approx a \frac{\sin(\frac{\pi a}{\lambda}\theta)}{\frac{\pi a}{\lambda}\theta} \implies \Delta\theta = \frac{\lambda}{a}$$

Single slit diffraction

-Pulse width-bandwidth relation

Application of Equation (8) to pulse train and gratings

-Grating for deflecting
A grating converts plane waves propagating in the z-direction into plane waves propagating at an angle following the grating equation.

-Spatial harmonics and angles of propagation (Section 4.2)
-
The spatial distribution of arbitrary function can be expanded into spatial harmonics of various frequencies. In the far field, which is sufficiently far away so that the plane waves are separated, a single plane wave contributed to the amplitude at a far away plane. This is the **Fraunhofer approximation**
.
Thus optics can be used to computer the Fourier transform of a spatial function. The spatial function can be program and generated using a liquid crystal light modulator.

-Graded grating for focusing

-Fresnel lens

Fourier transform between two focal planes of a lens

First we need to work out some basic procedures concerning propagation in free space.

Transfer function in free space for a spatial harmonics
Consider a two-dimensional complex electric field at z=0 given by

$$f(x, y) = Ae^{-j2\pi\nu_x x - j2\pi\nu_y y}$$
(11)

where the ν's are the spatial frequencies in the x and y directions. The spatial frequencies are the inverse of the periods.

When this pattern is emitting at wavelength λ, a plane wave of the following form is generated:

$$U(x, y, z) = Ae^{-j2\pi\nu_x x - j2\pi\nu_y y - jk_z z}$$
(12)

where

$$k_z = 2\pi(\frac{1}{\lambda^2} - \nu_x^2 - \nu_y^2)^{1/2}$$
(13)

This can be proven by considering the diffraction of a sinusoidal grating illuminated by a plane wave of wavelength λ.

Thus by decomposing a spatial distribution of electric field into spatial harmonics, each component can be treated separately.

Define a transfer function in free space for the spatial harmonics of spatial frequency ν_x and ν_y to travel from $z=0$ to $z=d$ as

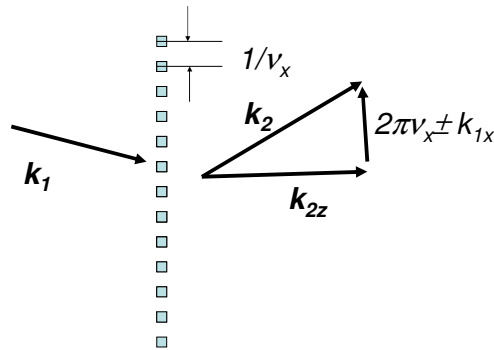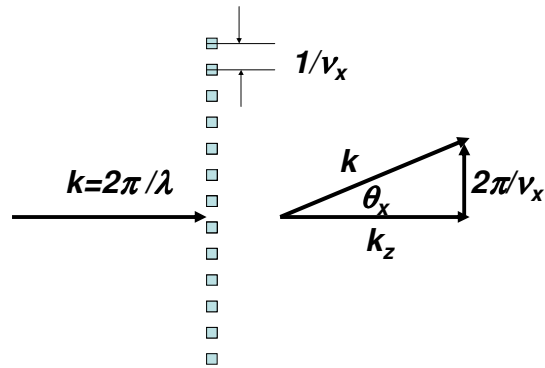$$H(\nu_x, \nu_y) = e^{-j2\pi(\frac{1}{\lambda^2} - \nu_x^2 - \nu_y^2)^{1/2}d}$$
(14)

This is the multiplication factor to a sinusoidal spatial pattern of spatial frequency ν_x and ν_y. The redirecting of an incident plane wave into another direction by a periodic structure can be illustrated in the following figure. If the spatial periodic structure is stationary, the outgoing wave number is the same as the incident wave number.

A simple rule to remember:

When a plane wave of wave vector k_1 is incident on a spatial harmonics, such as a grating with spacing d extending in the x-direction, the wave vector, k_2 of the outgoing wave can be related to k_1 by the following relation.

$$(\vec{k}_2 \pm \vec{k}_1) \bullet \hat{x} = 2\pi\nu_x N$$
(14.1)

Where N is an integer.

$1/v_x$

$k=2\pi/\lambda$

k

θ_x

k_z

$2\pi/v_x$

$1/v_x$

k_2

$2\pi v_x \pm k_{1x}$

k_1

k_{2z}

The H factor can be simplified in the limit of small spatial period compared to the wavelength. $(1/v \ll \lambda)$. (What is the meaning ?)

Using the Fresnel approximation (small angle

approximation) $2\pi(\frac{1}{\lambda^2} - v_x^2 - v_y^2)^{1/2} d = 2\pi \frac{d}{\lambda}(1-\theta^2)^{1/2} \approx 2\pi \frac{d}{\lambda}(1 - \frac{\theta^2}{2})^{1/2}$
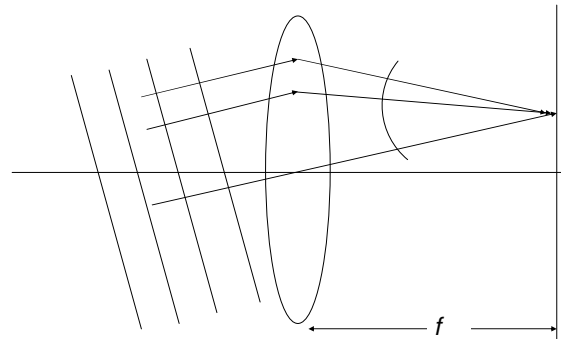
Thus (14) becomes

$$H(v_x, v_y) = e^{-jkd} e^{-j\pi\lambda d(v_x^2 + v_y^2)}$$

(15)

Thus the phase change as a function of propagation distance d in free space result in a phase change that is quadratic function of spatial frequency v's.

The effect of lenses

A converging lens transforms a plane wave into a spherical wave which converges toward a point at one focal length away.

Converging lens

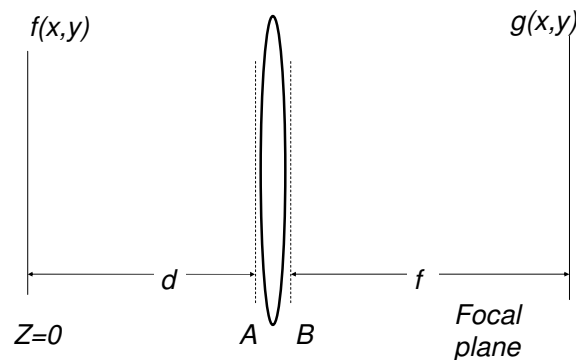A lens is to introduce a quadratic phase shift to the wavefront given by

$$. e^{j\pi \frac{x^2+y^2}{\lambda f}}$$

(16)

How to prove by considering the surfaces of constant phase. ?.

Examples for the application of the transfer functions

Fourier transform using a lens

An intuitive picture will be provided.

First the function $f(x,y)$ at z=0 can be expanded using the spatial harmonics:

$$f(x, y) = \int F(V_x, V_y) e^{-j2\pi(V_x x + V_y y)} dV_x dV_y \qquad (17)$$

The integrand is then propagate in free space to plane A, using (15). Then after the lens to plane B using (16)

$$U_B(x, y, z_B) = e^{-jkd} e^{j\pi \frac{x^2+y^2}{\lambda f}} e^{-j\pi\lambda d(V_x^2+V_y^2)} F(V_x, V_y) e^{-j2\pi(V_x x + V_y y)}$$

$$= A(V_x, V_y) e^{j\pi \frac{(x-x_0)^2+(y-y_0)^2}{\lambda f}}$$

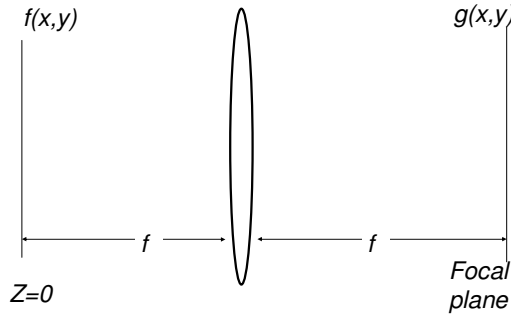where $A(V_x, V_y) = e^{-jkd} e^{j\pi\lambda(d-f)(V_x^2+V_y^2)} F(V_x, V_y)$, $x_0 = \lambda V_x f$, and $y_0 = \lambda V_y f$.

The waveform further propagates from plane B to the focal plane.

Results:

$$g(x, y) = h_f F(\frac{x}{\lambda f}, \frac{y}{\lambda f}) \qquad (18)$$

where $h_l = (\frac{j}{\lambda f}) e^{-j2kf}$

A lens has the effect of bring the far-field angular distribution into the focal plane.

Holography

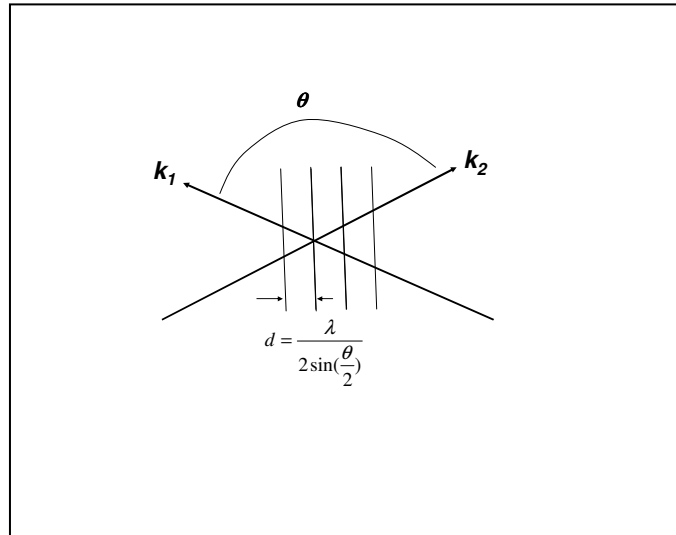*Interference fringes caused by two plane waves*

Two plane waves whose wave vector subtends an angle θ create interference fringe lines in the direction bisecting the angle and with a separation

$$d = \frac{\lambda}{2\sin(\frac{\theta}{2})} . \qquad\qquad (19)$$

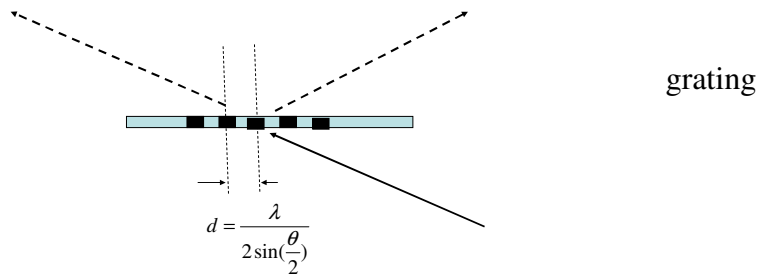Example: Counter-propagating beams create standing waves of spacing $\lambda/2$.

Figure 1

Thin planar holography
Show that a grating with the spacing of the fringes, Eq. (19) is illuminated by wave k_1, one of the diffracted waves is in the direction of k_2.

Figure 2

Actually, there may be two diffractive waves, corresponding to different orders.

grating

Bragg condition

Path difference of waves from successive layers is

$$\Delta = \overline{AB} + \overline{BC} - \overline{CD} = 2d\cos\theta \qquad\qquad (20)$$

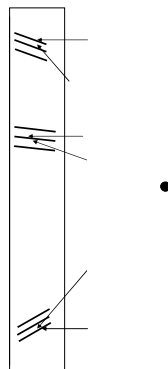Maximum reflection occurs when Δ equals the wavelength.

The Bragg reflectors are widely used as mirrors in laser systems and optical fibers. Typically, alternating layers of higher and lower indices of refraction are used.
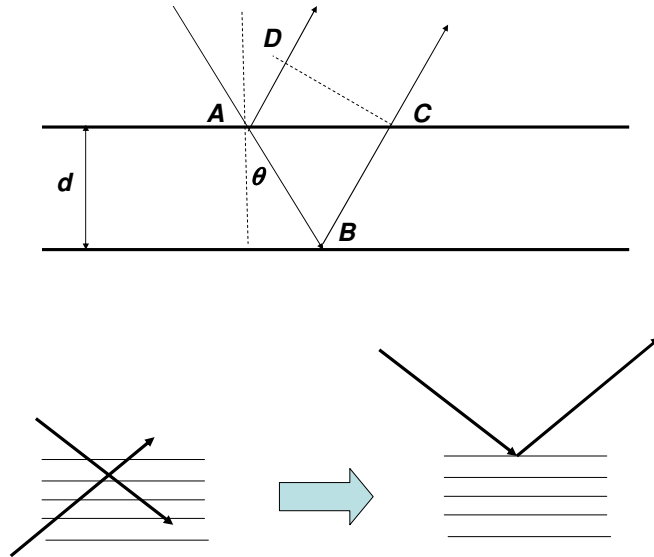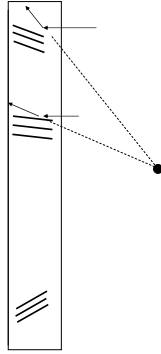

Volume holography

Example of image reconstruction of a point illuminated by a plane wave.

Recording

Reading

Consider am arbitrary object waves at (x,y,z) to be $E(x,z)e^{-j(k_xx+k_zz)}$ making an local angle θ at (x,y,z). Assuming that the reference wave is a plane wave E_re^{jkz} in the z-direction.

The superposition of the two result in interference fringes pattern. At z=0, the intensity is

$$\left|E_r + E(x,y)e^{-jk_xx}\right|^2 = E_r^2 + E_rE(x,y)e^{-jk_xx} + E_rE(x,y)e^{jk_xx} + E^2(x,y)$$

If the fringe pattern is recorded on a transparency and illuminated by plane waves propagating in the z-direction, the diffractive waves are

$$E_{diffracted} \propto E_r^2e^{-jkz} + E_rE(x,y)e^{-jk_xx-jk_zz} + E_rE(x,y)e^{jk_xx-jk_zz} + E^2(x,y)e^{-jkz}$$

The first the last terms are the transmitted reference waves, and the filtered intensity. The second term is the reconstructed the object waves and the third term is the conjugation of the object waves. Often the reconstructed and the conjugate waves form the image and virtual image.

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Problems:

1. Prove Eqs. (5)-(8). You don't need to turn in the answer, but must prove it once.
2. Prove (19).
3. The beam divergence at infinity of a plane wave passing through a single slit of width is known to be $\theta = \dfrac{\lambda}{D}$ where the angle is measured from the maximum intensity at the center to the first intensity minimum on either side, and D is the slit width for D>> λ.
   Use the transfer function approach to obtain the same relation.
   (a) First, express *F(x),* the x-dependency of the wave, after the slit using Fourier transform.
   (b) Apply the transfer function (15).

(c) Express the resultant function as a function of $\theta = \dfrac{x}{z}$ for z approaching infinity,

$F(x)$ $H(v_x)F(x)$

D

z=0 $z \to \infty$
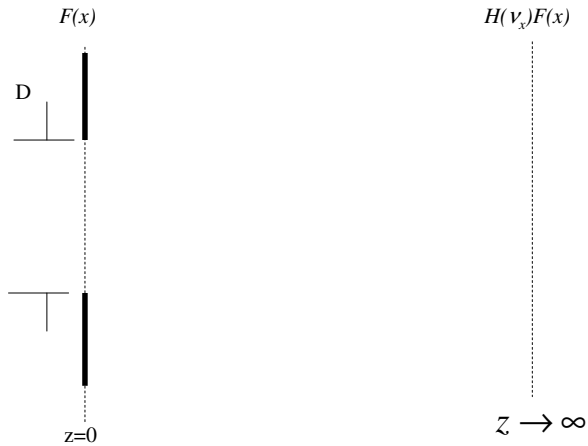
4. Find all possible directions of the diffracted beams when the planar grating, shown in Figure 2, is illuminated at an angle θ/2 from the normal. The grating is created by the interference of two waves shown in Figure 1.
5. Prove the Bragg condition Eq(20).
6. Design a holographic converging lens of focal length *f* for wavelength λ. Specify the fringe spacing as a function of position in the plane of the hologram.