5 Waves in dispersive media

Bound electron model of dielectric constant/index of refraction

The behavior of electrons when driven by an electromagnetic radiation can be understood using the well-known equation of forced oscillator, in terms of the mass, oscillation frequency, and damping coefficient:

$$\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + \omega_0^2 x = \frac{e}{m}E \tag{1}$$

where x is the displacement vector, $\omega_0 = (\kappa/m)^{1/2}$ and σ is a damping constant.
The polarization density of the medium is the sum of the dipole moments of N-atoms per unit volume so that $P = Nex$. Eq (1) becomes

$$\frac{d^2P}{dt^2} + \sigma\frac{dP}{dt} + \omega_0^2 P = \omega_0^2\varepsilon_0\chi_0 E \tag{2}$$

where $\chi_0 = e^2N/m\varepsilon_0\omega_0^2$ is the susceptibility.

For a monochromatic electric field of frequency ω,

$$(-\omega^2 + j\omega\sigma + \omega_0^2)E = \omega_0^2\varepsilon_0\chi_0 E$$

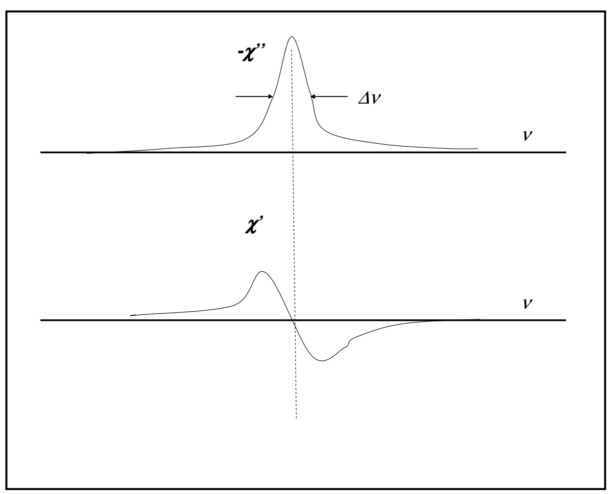
$$\chi(\nu) = \chi_0\frac{\nu_0^2}{\nu_0^2 - \nu^2 + j\nu\Delta\nu}$$

where the bandwidth $\Delta\nu = \frac{\sigma}{2\pi}$. The susceptibility is a complex number. The meaning of damping?

The real and imaginary parts are

$$\chi' = \chi_0\frac{\nu_0^2(\nu_0^2 - \nu^2)}{(\nu_0^2 - \nu^2)^2 + (\nu\Delta\nu)^2} \tag{3}$$

$$\chi'' = -\chi_0\frac{\nu_0^2\nu\Delta\nu}{(\nu_0^2 - \nu^2)^2 + (\nu\Delta\nu)^2} \tag{4}$$



If the atoms are placed in a medium of index of refraction n_0,

$$n(\nu) = n_0 + \frac{\chi^{'}(\nu)}{2n_0} \quad (5)$$

$$\alpha(\nu) = -\left(\frac{2\pi\nu}{n_0 c_0}\right)\chi^{''}(\nu) \quad 6)$$

The imaginary part of the susceptibility can lead to gain or absorption loss.

Pulse propagation in dispersive medium

Consider a plane-wave pulse *U(z,t)* propagating in the z-direction, the propagation of the pulse may be analyzed by treating the traveling of the individual frequency components.

$$U(z,t) = A(z,t)\exp(j(2\pi\nu_0 t - \beta_0 z) \quad (7)$$

where $\beta_0 = \beta(\nu_0)$ is the central wavenumber and A is the complex envelop of the pulse which is slow-varying. This is a wave packet of central frequency ν_0. The propagation can be treated by considering the frequency components of the wave at the initial point z=0.

$$A(0,t) = \int_{-\infty}^{\infty} a(0,f)\exp(j2\pi ft)df \quad (8)$$

and amplitude for the frequency f is given by

$$a(0,f) = \int_{-\infty}^{\infty} A(0,t)\exp(-j2\pi ft)df$$

Here it is assumed that f is the frequency deviation from the central frequency and $f << \nu_0$. The "frozen" wave at z=0 in (7) then is

$$U(0,t) = \int_{-\infty}^{\infty} a(0,f)\exp(j2\pi ft)\exp[+j2\pi\nu_0 t]df \quad (9)$$

By expanding $\beta(\nu)$ surrounding ν_0, the wave then travel to z according to

$$U(z,t) = \int_{-\infty}^{\infty} a(0,f)\exp(j2\pi ft)\exp[-j\beta(\nu_0 + f)z]\times\exp[j2\pi\nu_0\,]df$$

$$= \int_{-\infty}^{\infty} a(0,f)exp[j2\pi(\nu_0 + f)t]\exp(-jf\frac{d\beta}{d\nu}z)\exp(-j\frac{1}{2}f^2\frac{d^2\beta}{d\nu^2}z)df \quad \exp(-j\beta_0 z) \quad (10)$$

$$= \int_{-\infty}^{\infty} a(0,f)\exp(j2\pi[(\nu_0 + f)t])\exp(-j2\pi f\frac{z}{V_g})\exp(-j\pi Dzf^2)\exp(-j\beta_0 z)df$$

where the group velocity $V_g = \frac{2\pi}{\frac{d\beta}{d\nu}}$ and dispersion coefficient $D = \frac{1}{2\pi}\frac{d^2\beta}{d\nu^2}$. (11)

Case I Dispersion free medium
For D=0, (10) becomes

$U(z,t) =$

$$= \int_{-\infty}^{\infty} a(0,f)\exp(j2\pi[(\nu_0 + f)t])\exp(-j2\pi f \frac{z}{V_g})\exp(-j\beta_0 z)df$$

$$=[\int_{-\infty}^{\infty} a(0,f)\exp[j2\pi f(t-\frac{z}{\nu_g})df]\exp(j2\pi\nu_0 t - j\beta_0 z)$$

The leads to an envelop function $A(t-\frac{z}{v_g}, z)$ which is centered at $\nu_g t$ to without changing shape.

Assuming that the initial pulse shape A(0,t) has a Gaussian profile of width τ_0:

$$A(0,t) = A_0 \exp(-\pi \frac{t^2}{\tau_0{}^2})$$

Substituting $a(0,f) = \exp[-\pi(f\tau_0)^2]$ into (10)

$$U(z,t) = \int_{-\infty}^{\infty} \exp[-\pi(f\tau_0)^2]\exp(j2\pi[(\nu_0 + f)t])\exp(-j2\pi f \frac{z}{V_g})\exp(-j\pi Dzf^2)\exp(-j\beta_0 z)df$$

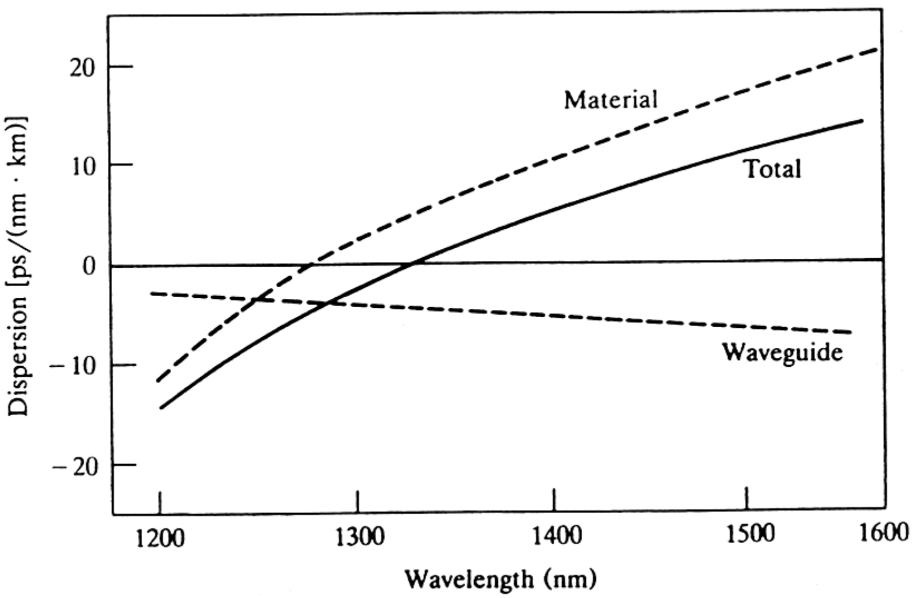
$$= \{\int_{-\infty}^{\infty} \exp[(j2\pi f(t-\frac{z}{\nu_g})]\exp[-\pi(f\tau^{'})^2]df\} \times \exp(j2\pi\nu_0 - j\beta_0 z) \tag{12}$$

where $\tau'^2 = \tau_0{}^2 + jDz$. After the integration, the electric field has the following form

$$|U(z,t)| \propto \exp(-\pi \frac{(t - z/v_{g^2})^2}{\tau^2}) \tag{13}$$

where $\tau^2 = \tau_0^2 + \frac{D^2 z^2}{\pi^2 \tau_0^2}$ (14)

The integration of Eq (12) is simply Eq (8), shifted to a new location $z = \nu_g t$ and with a Gaussian shape of width τ , which is a complex number.



Maintaining pulse shape in a dispersive medium by frequency chirping

In Eq. (12), if the phase of the original Gaussian pulse is phase modulated during the pulse by a factor $\exp(+j\pi Dzf^2)$ to cancel the broadening effect, the pulse duration may be maintained while propagating in a dispersive mediu--optical soliton

Problem 5.5-2

Problem 5.6-1

Pulse broadening in optical fibers 5.6-3