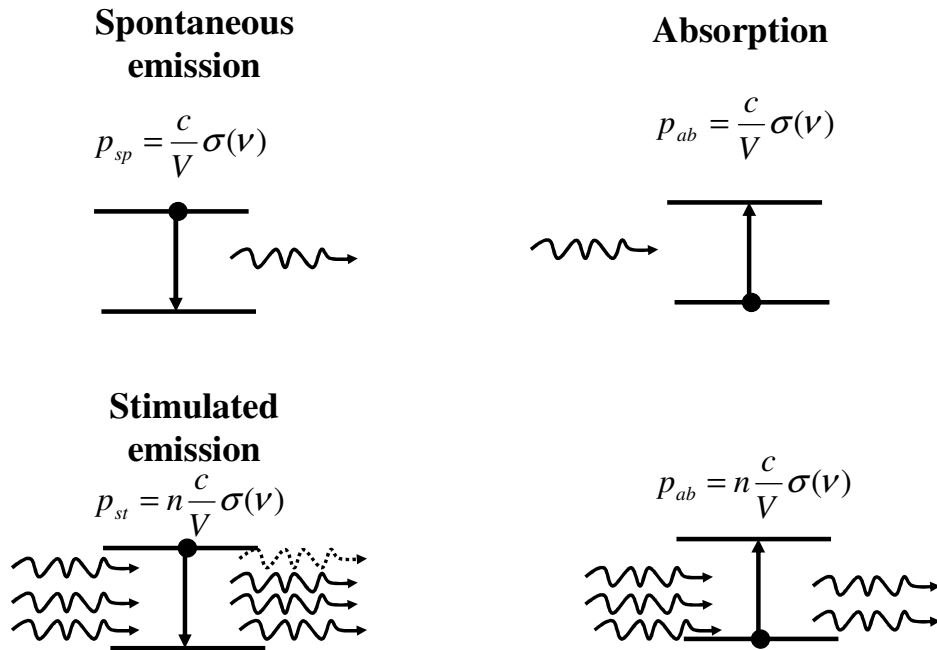# Laser amplifier

**A review of quantum mechanical result of emission and absorption**

Symmetric relation between absorption and spontaneous emission.

**Spontaneous emission**

$$p_{sp} = \frac{c}{V}\sigma(v)$$

**Absorption**

$$p_{ab} = \frac{c}{V}\sigma(v)$$

**Stimulated emission**

$$p_{st} = n\frac{c}{V}\sigma(v)$$

$$p_{ab} = n\frac{c}{V}\sigma(v)$$

**Interaction of single mode light with a single atom**

*Spontaneous emission: stimulated emission with zero incident photon*

In a cavity of volume $V$, the probability density or rate ( per second) of a (**single atom** ) spontaneous emission into a single mode is

$$p_{sp} = \frac{c}{V}\sigma(v) \tag{1}$$

where $\sigma(v)$ is the transition cross section, which is a narrow line function centered at $v_0$. The probability of decay between t and $t+\Delta t$ is $p_{sp}\Delta t$. The inverse of $p_{sp}$ is the lifetime of the excited state emitting in a particular mode only. The actual lifetime of atomic system is much shorter because there are many modes to emit into.

*Absorption*

The probability density for the absorption of a photon from a given mode of frequency $v$ in a cavity of volume $V$ is governed by the same rule

$$p_{ab} = \frac{c}{V}\sigma(v) \tag{2}$$

If there are n photons present in one mode, the probability density that the atom absorbs one photon is $n$ times larger or

$$p_{ab} = n\frac{c}{V}\sigma(v) \tag{3}$$

*Stimulated emission*
The probability of creation of photons when only one photon present is govern by the same cross section as (1)

$$p_{st} = \frac{c}{V}\sigma(v) \tag{4}$$

If there are already n photons in a mode, the probability of emitting one more photon is

$$p_{ab} = n\frac{c}{V}\sigma(v) \tag{5}$$

Thus the $p_{st}$ is the same as $p_{ab}$, the probability for spontaneous and stimulated emission of one photon is the same. Since the spontaneous emission occurs in addition to the stimulated emission, the total probability of an atom emitting into one mode in the presence of $n$ photons is $(n+1)\frac{c}{V}\sigma(v)$, while the total probabilyt of absorbing one photon in the presence of $n$

photons $n\frac{c}{V}\sigma(v)$

*Oscillator strength, S, (cm$^2$s$^{-1}$)* is the area of the probability density as a function of frequency

$$S = \int_0^\infty \sigma(v)dv \tag{6}$$

Line shape function $g(v)$ is a normalized function of $\sigma(v)$
$$\sigma(v) = Sg(v) \tag{7}$$

**Spontaneous emission probability density (into all modes)**

$$P_{ab} = \int \left|\frac{c}{V}\sigma(v)\right| VM(v)dv = c\int \sigma(v)M(v)dv \tag{8}$$

where $M(\nu) = 8\pi\nu^2/c^3$ is the modal density at frequency $\nu$ in a unit volume.   Since $M$ is nearly constant at optical frequencies centered at $\nu_0$,

$$P_{ab} = M(\nu_0)cS = \frac{8\pi S}{\lambda^2} \qquad (9)$$

The lifetime of the spontaneous emission is defined as

$$\frac{1}{t_{sp}} = P_{ab} = \frac{\lambda^2}{8\pi S} \qquad \textbf{(10)}$$

The oscillator strength is

$$S = \frac{\lambda^2}{8\pi t_{sp}} \qquad \textbf{(10)-1}$$

**Stimulated emission and absorption with many photons of a single frequency (monochromatic) propagating in one direction**

Let $\phi$ be the photon flux density at frequency $\nu$, and I be the intensity

$$\phi = \frac{I}{h\nu} \quad \text{(unit: photons/ cm}^2\text{/s)}$$

When the stream of photons interact with an atom, the probability density for stimulated emission and absorption is $W_i = P_{ab} = P_{st}$

Consider the problem of the interaction between an atom and photons in a volume $V$.  Assume that the volume is a cylinder stretching along the direction of propagation by the length of $c$ x1 second.
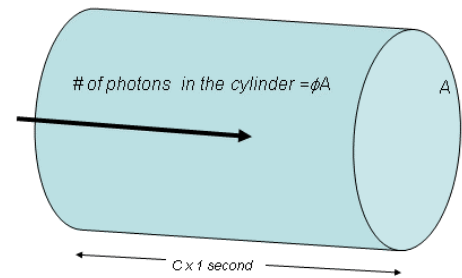The number of photons in the cylinder is

$$n = \phi A = \phi\frac{V}{c} \qquad (11)$$

Thus the probability for  a photon flux interaction with  a single atom, from eq. (4)  is given by

$$W_i = n(c\frac{\sigma}{V}) = \phi\sigma \qquad (12)$$



# of photons  in the cylinder $=\phi A$

$A$

$C$ x 1 second

W is a dimensionless quantity.

If the incident photon flux has a broad bandwidth, the overall probability of absorption and stimulated emission is

$$W_i = \int_0^\infty \frac{\rho(\nu)V}{h\nu}\left[(c\frac{\sigma}{V})\right]d\nu \approx \frac{\rho(\nu_0)}{h\nu_0}cS = \frac{\lambda^3}{8\pi h t_{sp}}\rho(\nu_0) = \frac{\bar{n}}{t_{sp}} \tag{13}$$

where $\rho(\nu)$ is the spectral energy density per unit bandwidth per unit volume for the incident photons and $\bar{n}$ is the average photons per mode. The probability of stimulated emission is enhanced by n times in the presence of photon flux.

Note that $W_i$ is the same for stimulated emission and absorption.

**Of historical interest: Einstein's A and B coefficients**

Einstein obtained the relation using a different approach, by considering the interaction of photons and atoms in thermal equilibrium.

Consider the interaction between thermal radiation with atoms with energy levels at $E_1$, $E_2$, where $E_2 > E_1$. The ratio of the numbers of atoms in each state is

$$\frac{N_1}{N_2} = e^{-(E_1-E_2)/kT} \tag{14}$$

The number of spontaneous emission from atoms at $E_2$ is $N_2 A_{21}$ where $A_{21}$ is the transition probability per unit time for transition from 2 to 1 and is know as the A coefficient.

Let $B_{12}$ and $B_{21}$ denote the proportionality constants for stimulated emission. Then the numbers of stimulated downward transition (emission) and upward (absorption) per second in the presence of photon density $u$ are $N_2 B_{21} u$ and $N_1 B_{12} u$. The B's are the B coefficients.

Under thermal equilibrium, the following relation should hold:
$$N_2 A_{21} + N_2 B_{21} u = N_1 B_{12} u \tag{15}$$

By solving for $u$:
$$u = \frac{N_2 A_{21}}{N_1 B_{12} - N_2 B_{21}} \tag{16}$$

Using (14), we can write
$$u = \frac{A_{21}}{B_{21}} \frac{1}{(B_{12}/B_{21})e^{h\nu/kT} - 1} \tag{17}$$

In order to agree with the plank radiation formula, the following equations must hold:
$$B_{12} = B_{21}$$
$$\frac{A_{21}}{B_{21}} = \frac{8\pi h \nu^3}{c^3} \tag{18}$$

Eq(18) governs the relation between the A and B coefficients.

The Einstein's coefficients can be related to the results based on the probability arguments by

$$A = \frac{1}{t_{sp}} \tag{19}$$

$$B = \frac{\lambda^3}{8\pi h t_{sp}} \tag{20}$$

Thus the probability in Eq (15) becomes $W_i = B\rho(v_0)$ (21)

**Line Broadening**

Discussions on homogenous broadenings vs inhomogeneous broadenings

*Homogenous broadening— the broadenings shared by all atoms of a medium*

(1) Lifetime broadening --caused by the spread of the energies of the terminal states due to their finite lifetime

$$\Delta E = \frac{h}{2\pi}\frac{1}{\tau} = \frac{h}{2\pi}(\frac{1}{\tau_1} + \frac{1}{\tau_2}) \tag{22}$$

where $\tau_1$ and $\tau_2$ are the lifetime of the upper and lower states and $\tau$ is the effective lifetime. The line shapes of these lifetime broadened transitions are Lorentzian given by

$$g(v) = \frac{\Delta v / 2\pi}{(v - v_0)^2 + (\Delta v / 2)^2} \tag{23}$$

A simple way of explaining the line shape based on single electron damped oscillator driven by an sinusoidal electric field.

Next we will express the transition cross section defined earlier with the line shape function. From (6) and (7), the Oscillator Strength, S, the transition cross section,$\sigma(v)$, and line shape function are defined by

$$S = \int_0^\infty \sigma(v)dv \tag{6) (24}$$

$$\sigma(v) = Sg(v) \tag{7)(25}$$

where, from (10-1), $S = \frac{\lambda^2}{8\pi t_{sp}}$. Combinging (24) and (25) we have

$$\sigma(v) = \frac{\lambda^2}{8\pi t_{sp}} g(v) \tag{26}$$

The peak cross section at the center frequency of the line $v_0$ is

$$\sigma_0 = \frac{\lambda^2}{2\pi}\frac{1}{2\pi t_{sp}\Delta v} \tag{27}$$

For a special case in which the linewidth $\Delta \nu$ is entirely caused the lifetime $t_{sp}$ of the excited state,

$$\sigma_0 = \frac{\lambda^2}{2\pi}$$ 
(28).

In this case, for $\lambda = 1$ μm, the peak cross section is $10^{-9}$ cm$^2$ . The actually cross section for most solid-state laser materials is on the order of $10^{-20}$ cm$^2$ because the linewidth are much wider than that caused by the lifetime of the excited state only.
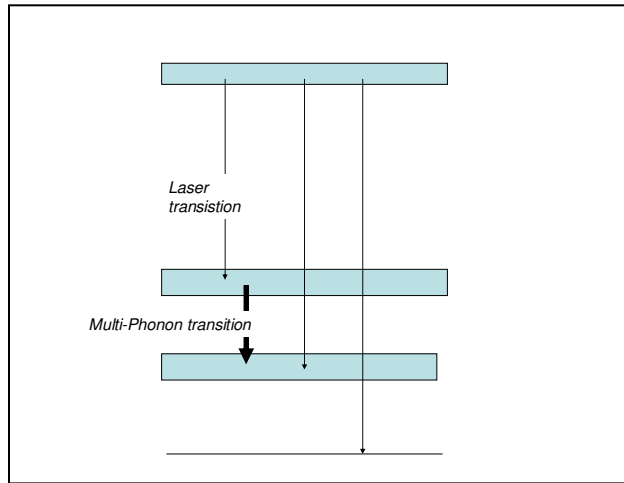


(2) Collision broadening in gas and lattice vibration in solid

*Inhomogenous broadening*

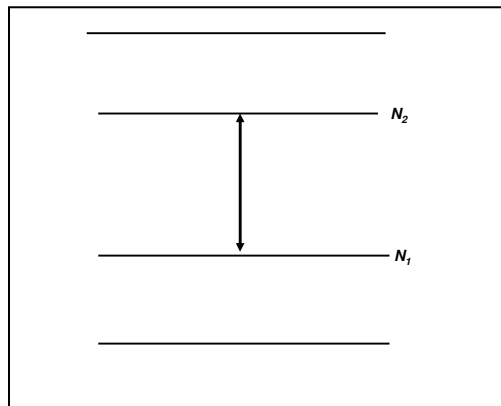Doppler effect in gas— The typical lineshape of a Doppler-broadened line is not Lorentizian, but **Gaussian.**
Different local field in crystals and glasses

*Hole-burning*

**Laser amplifiers**

A brief discussion of key features of an amplifier
1. Gain
2. Bandwidth
3. Phase shift
4. Power source (pumping method)
5. Gain saturation
6. Noise

(1) *Amplifier gain*

From (12), the probability that an atom in the ground state absorbing a photon is

$$W_i = n(c\frac{\sigma}{V}) = \phi\sigma(\nu) \tag{12)(29}$$

where, from (26), $\sigma(\nu) = \dfrac{\lambda^2}{8\pi t_{sp}} g(\nu)$. When there are $N_1$ atoms in the lower level and $N_2$

atoms in the upper level , the average number of absorbed photons is $N_1 W_i$ , and the number of produced photons is $N_2 W_i$.  The net production of photons is given by $(N_2 - N_1)W_i = NW_i$, where N is the population density difference between the upper and lower states.
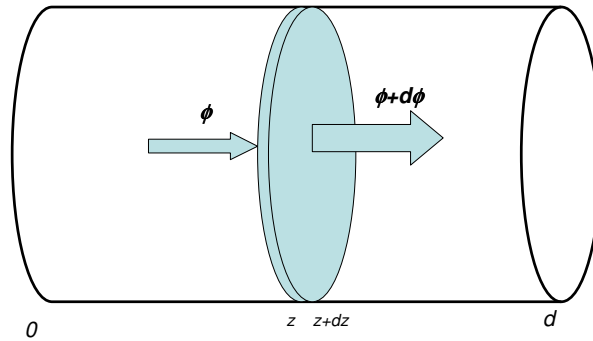


The increment of photon flux passing through a slice of gain medium is

$$d\phi = NW_i dz \tag{30}$$

Define the  gain coefficient of a  laser medium as

$$\gamma(\nu) = N\sigma(\nu) = N\frac{\lambda^2}{8\pi t_{sp}} g(\nu) \tag{31}$$

The photon flux along the medium follows the following equation:

$$\frac{d\phi(z)}{dz} = \gamma(\nu)\phi(z) \tag{31}$$

The solution of  (31)  is an exponentially increasing function of z.

$$\phi(z) = \phi(0)\exp[\gamma(\nu)z] \tag{32}$$

To exhibit a net gain, *N* must be larger than zero, the condition known as population inversion.

The above treatment is also applicable to the absorption process when *N<0*.

The overall gain of the laser amplifer of length *d* is

$$G(\nu) = \exp[\gamma(\nu)d] \tag{33}$$

(2) Amplifier bandwidth

The amplifier bandwidth, measured in units of frequency or wavelength, is simply the width of the lineshape. For a Lorentzian shape as defined in (23), the bandwidth is $\Delta v$. The gain coefficient is also Lorentian
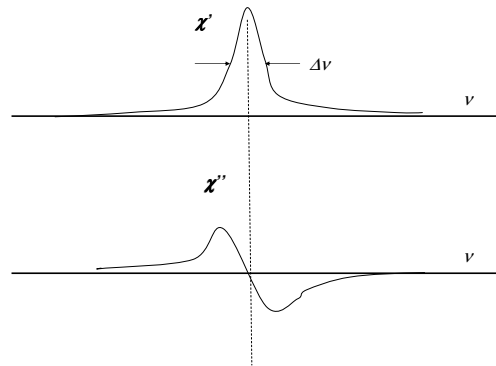
$$\gamma(v) = \gamma(v_0)\frac{(\Delta v/2)^2}{(v-v_0)^2 + (\Delta v/2)^2} \tag{34}$$

where $\gamma(v_0)$ is the gain at the center of the line.

How does the gain bandwidth affect the amplification?
- Tunability
- Pulse duration

(3) Phase shift



We will use the result of Lecture 5 for the bound-electron model to show that the real and imaginary parts of the susceptibility are related.

$$\chi' = \chi_0 \frac{v_0^2(v_0^2 - v^2)}{(v_0^2 - v^2)^2 + (v\Delta v)^2}$$

$$\chi'' = -\chi_0 \frac{v_0^2 v \Delta v}{(v_0^2 - v^2)^2 + (v\Delta v)^2}$$

The real part contributed to the index of refraction which in turns creates a phase shift of the wave. The imaginary part is the gain or absorption loss. The relation between the absorption and phase shift is the Kramers-Kronig relation.

$$\chi'(v) = \frac{v - v_0}{\Delta v}\chi''(v) \tag{35}$$

Thus a laser amplifier creates a net change in the amplitude of the field and a phase change. is a complex number. The phase change is related to the gain by

$$\varphi(v) = \frac{v - v_0}{\Delta v} \gamma(v) \tag{36}$$
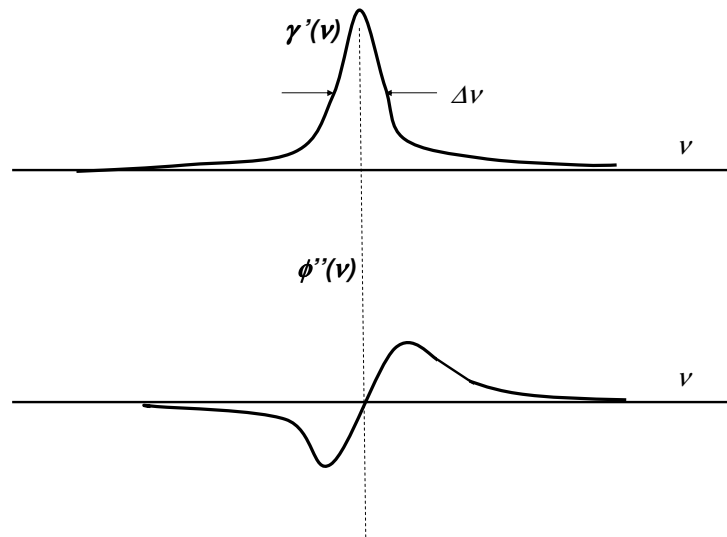


The electric filed passing through a slab of thickness dz is governed by the equation

$$\frac{dE}{dz} = E(z)[\frac{1}{2}\gamma(v) - j\varphi(v)] \tag{37}$$

Thus, if the input signal has a broad bandwidth the covers the entire gain bandwidth, the amplified signal on the low frequency side has a different phase as the high frequency side, resulting in distortion in pulse shape, broadening or compression.

(4) Amplifier power source

**A four-level system**
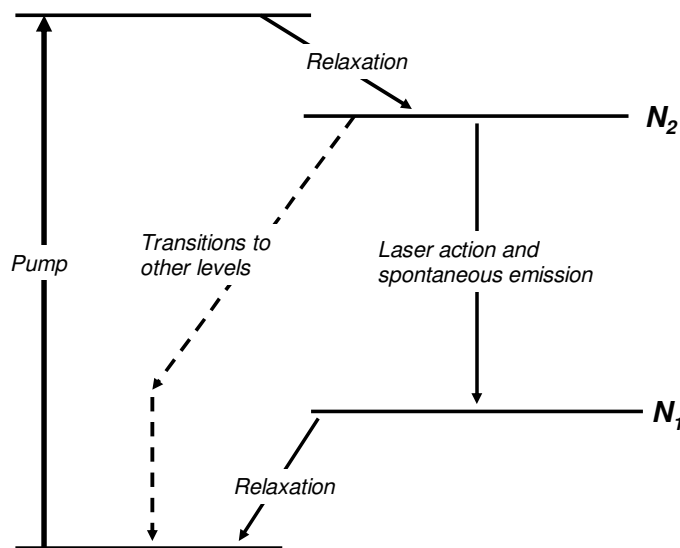
## Rate equations in the absence of amplifier radiation

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2}$$

$$\frac{dN_1}{dt} = -R_1 - \frac{N_1}{\tau_1} + \frac{N_2}{\tau_{21}}$$

(38)

where $R_2$ is the rate of pumping into level 2 and $R_1$ is the rate of pumping out of level 1. The steady state (time derivative =0) population difference $N_0$ is

$$N_0 = R_2\tau_2(1 - \frac{\tau_1}{\tau_{21}}) + R_1\tau_1$$

(39)

*Discussions on how various factors affect the gain in two-level, three-level, four level systems.*

## Rate equations in the presence of amplifier radiation

$$\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} - N_2W_i + N_1W_2$$

$$\frac{dN_1}{dt} = -R_1 - \frac{N_1}{\tau_1} + \frac{N_2}{\tau_{21}} + N_2W_i - N_1W_i$$

(40)

The steady-state solution is

$$N = \frac{N_0}{1 + \tau_s W_i}$$

$$\tau_s = \tau_2 + \tau_1(1 - \frac{\tau_2}{\tau_{21}})$$

**(41)**

The population difference between the upper and lower levels decreased in the presence of stimulated emission, a phenomenon known as **gain saturation**. The time $\tau_s$ is the saturation time constant.

Small signal regime: $\tau_s W_i << 1$. *–The depletion can be neglected.*

*Discussions: The effect of long and short lifetime of the upper and lower levels on on the saturation.*

## Gain coefficient

Combining (41) (31) and (29), the gain coefficient can be expressed using the unsaturated gain coefficient $\gamma_0(\nu)$, the saturation flux and the saturation photon flux,

$$\gamma(\nu) = \frac{\gamma_0(\nu)}{1 + \phi/\phi_s(\nu)}$$

(42)

where $\phi$ is the photon flux and

$$\gamma_0(\nu) = N_0 \frac{\lambda^2}{8\pi\tau_{sp}} g(\nu)$$

(43)

$$\frac{1}{\phi(v)} = \frac{\lambda^2}{8\pi} \frac{\tau_s}{t_{sp}} g(v)$$ (44)

*Discussion of inhomogenous broadening and hole burning*

(5) Amplifier noise- (A qualitative discussion )

$$\frac{d\phi}{dz} = \gamma(v)\phi + \xi_{sp}(v)$$ (45)

$$\xi_{sp} = N_2 \frac{1}{t_{sp}} g(v)B\frac{d\Omega}{8\pi}$$ (46)

where $\xi$ is the flux of the spontaneous emission to be amplified together with the "signal, B is the frequency bandwidth, and $d\Omega$ is the solid-angle of acceptance. The physical meaning is quite straightforward.

**Exercises**

1. 13.3-2     (p.482)   about the broadening of the saturated gain profile.
2. Problem 13.2.1   (p.492)
3. Problem 13.3.2
4. Problem 13.3.3